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are Roots

A geometric approach using exact square manipulatives can promote an understanding of the algorithm that can dismantle radical expressions.

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"You don't understand anything until you learn it more than one way."
Marvin Minsky, cognitive scientist, cofounder of the Massachusetts Institute of Technology (MIT) Artificial Intelligence Laboratory

A picture is worth a thousand words. This statement is especially true in mathematics teaching and learning. Visual representations such as pictures, diagrams, charts, and tables can illuminate ideas that can be elusive when displayed in symbolic form only. The prevalence of representation as a mathematical process in such documents as *Principles and Standards for School Mathematics* (NCTM 2000), the Common Core State Standards for Mathematics (CCSSI 2010), and the Strands of Mathematical Proficiency (NRC 2001) reinforces the importance of mathematics teachers and their students using multiple representations when exploring mathematical ideas.

Simplifying square roots and other radicals is a staple of prealgebra and algebra 1 courses because it is a requisite skill for studying many other topics in the high school curriculum (CCSSI 2010; 8.EE.2). To simplify a radical expression, students must analyze factors of the radicand and determine whether any subset of these factors can be simplified with respect to the radical's index. For example, if the radical represents a square root (index 2), students determine if any of the radicand's factors are perfect squares. If so, the radical can be written in a simpler form. A basic example of this process using the traditional algebraic representation is shown below.

$$\begin{aligned}\sqrt{18} &= \sqrt{9 \cdot 2}\\ &= \sqrt{9}\sqrt{2}\\ &= 3\sqrt{2}\end{aligned}$$

In developing a lesson on this topic for algebra 1 students, we sought to address a variety of learning styles. Although we developed verbal and written strategies to supplement teaching the algebraic representation, we struggled initially to determine a way of representing a visual or geometric approach for simplifying radicals. After examining geometric representations of perfect square numbers (1, 4, 9, 16, . . .), we developed a way to think about more complicated examples (e.g., 18, 45, 72), creating tactile and virtual manipulatives that enabled students to explore radical simplification and other concepts.

We wanted our students to initially use this representation and accompanying manipulatives to demonstrate their thinking about radicals and their simplification. Through repeated use of the representation with concrete examples and through a growing familiarity with the concept of square root, our students should become accustomed to using the representation as a tool when thinking about more formal and generalizable examples. In this way, the representation will evolve from a tool *of* thinking to a tool *for* thinking (Fosnot and Dolk 2002). An analogy would be how young students initially use numbers to symbolize collections of objects (8 apples, 3 bears, 7 days, and so on). Eventually, however, the numbers themselves become tools for carrying out more abstract purposes (2 + 9 = 11).

We share our visual representation for simplifying radicals and our initial efforts to use this representation with middle school and high school mathematics students as well as prospective mathematics teachers.

Fig. 1 Blue counters representing 9 can be arranged in a square array; pink counters representing 8 cannot. Three counters, seen along any one of the square's four sides, represent the square root of 9.

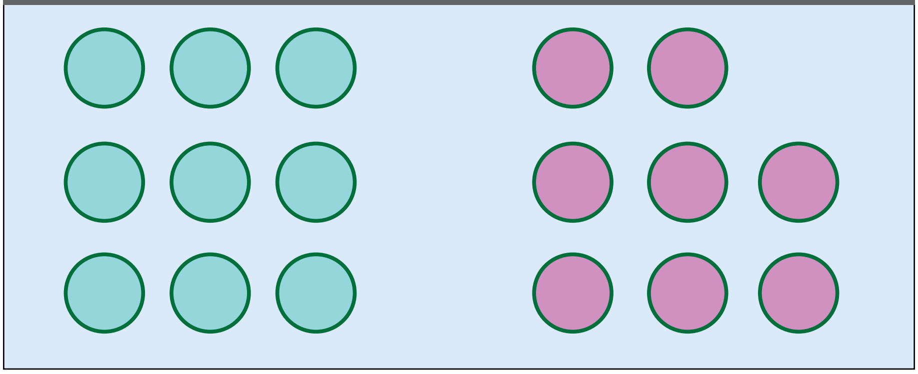
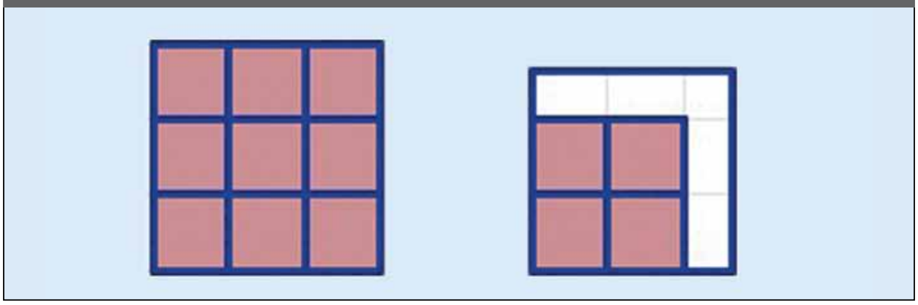


Fig. 2 A square with an area of 9 unit^2 can be partitioned into unit squares, but a square with an area of 7 units^2 cannot.



PERFECT SQUARE NUMBERS

To help students understand the concept of perfect square or square numbers, teachers can challenge students to arrange a collection of n objects (counters, blocks, pennies, beans, and so on) in a square-shaped array. Students can see that the objects representing some numbers can be arranged as a square array, whereas others cannot (see **fig. 1**). These square-forming numbers are *perfect squares*. Furthermore, students can determine the square root of a perfect square represented in this way by counting the number of objects along one edge of the square.

Similar to using an array of objects, students can represent perfect squares with an area model. The area of the square represents n; the square's side length is $\sqrt{n}$. Traditionally, such a model is used when $\sqrt{n}$ has a whole-number value. This model is viable because it can be partitioned into a square-shaped array of smaller 1 unit × 1 unit squares of area 1 unit^2.

A nonperfect square number such as 7, however, cannot be partitioned in this way. Although one can identify four 1 unit × 1 unit squares within the square, the excess space within the square cannot be partitioned further in this way (see **fig. 2**). At this point, a discussion of an area-based geometric representation of irrational numbers, such as $\sqrt{7}$, typically falls apart. By shifting our focus away from squares with rational side lengths (such as 1 unit), however, we have found a way to represent the process of simplifying more complicated radicals. This simplification illuminates the underlying math of the algebraic representation and algorithm that are traditionally taught to and used by students.

AN AREA-BASED GEOMETRIC REPRESENTATION

To illustrate our area-based geometric representation of simplifying radicals, consider $\sqrt{18}$. Similar to the square with area 7 square units, a square with area 18 square units cannot be partitioned into a square-shaped array of unit squares (see **fig. 3a**). Therefore, 18 is not a perfect square. This square, nevertheless, can still be partitioned into a square-shaped array of smaller squares, provided that the *area* of these smaller squares is rational even if the side lengths are irrational. In this case, the square with an area of 18 square units can be partitioned into 9 squares, each with an area of 2 square units (see **fig. 3c**). This new configuration provides an alternative way to calculate $\sqrt{18}$ by examining the length of the side of a square whose area is 18 square units. Notice that the side length of the large square is equivalent to three side lengths of one smaller square. The area of each small square is 2 square units, so the side length of each small square is $\sqrt{2}$ units. Thus,

$$\sqrt{18} = 3\sqrt{2}.$$

As a way to connect this model to the numeric solution above, the search for a perfect-square factor of 18 coincides with identifying how to partition the square into smaller squares with whole-number areas. The expression $\sqrt{9 \cdot 2}$ is equivalent to the length of the side of a square array of 9 squares each with an area of 2 square units. In this case, $\sqrt{2}$ is the length of a partitioned side of the large square into the nine small squares.

Figure 4 displays a similar approach used to simplify $\sqrt{75}$. Increasing the squared factor of the radicand increases the number of small squares in the corresponding area model. As a consequence, teachers should carefully select the examples they use in class because some figures may become rather intricate.

Fig. 3 A square with an area of 18 units^2 and small squares with a rational area of 2 units^2 help students see why $\sqrt{18} = 3\sqrt{2}$.

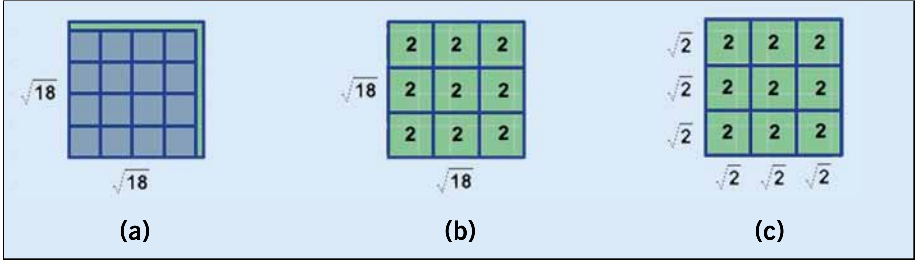
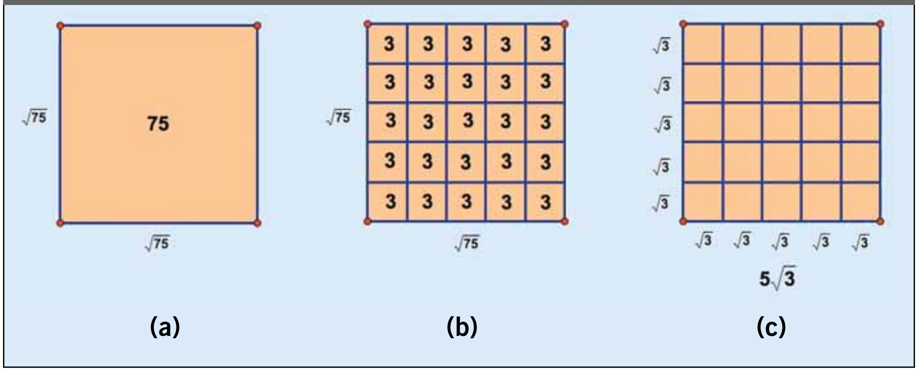


Fig. 4 Starting with a square with an area of 75 units^2, $\sqrt{75}$ is simplified using smaller squares, each with an area of 3 units^2.



THE GEOMETRIC REPRESENTATION IN USE

We were eager to have students try to simplify square roots using an area-based geometric representation. As we prepared activities, we considered the prerequisite knowledge needed to use the concrete manipulatives we had created and to transition eventually to abstract procedures. The key concept is the relationship between the numerical values of a square's area and side length. To ensure that students firmly grasped this geometric connection, we spent time at the start of the activities discussing this relationship.

In considering the best way for students to explore the geometric representation of square roots, Stephen Bismarck created a set of square manipulatives with whole number areas, dubbed *exact squares*, which students could arrange in configurations like those in **figures 3** and **4**. Students could select square tiles of a particular size to attempt to create squares of a larger area, physically modeling the process of finding a perfect square factor while sitting at their desks.

In addition, we created activities designed to help students connect their use of the geometric representation to the corresponding symbolic representation. We wanted to see how students at different stages of mathematical study would react to the concept presented through the manipulatives. We chose to use the Exact Squares activity with middle school students who had not been introduced to the symbolic process for simplifying square roots, precalculus high school students, and third-year preservice mathematics teachers. By working with students at various levels, we could focus on how students connected the geometric model to the traditional algorithm.

Each Exact Squares activity lasted about ninety minutes, comprising one class for the high school and college students and two classes for the middle school students. Students worked in small groups (three to four) and used activity packets and one set of precut exact squares (1 cm^2 through 10 cm^2).

Fig. 5 Casey, a prospective teacher, describes the geometric significance of each quantity used in the symbolic algorithm for simplifying $\sqrt{12}$.

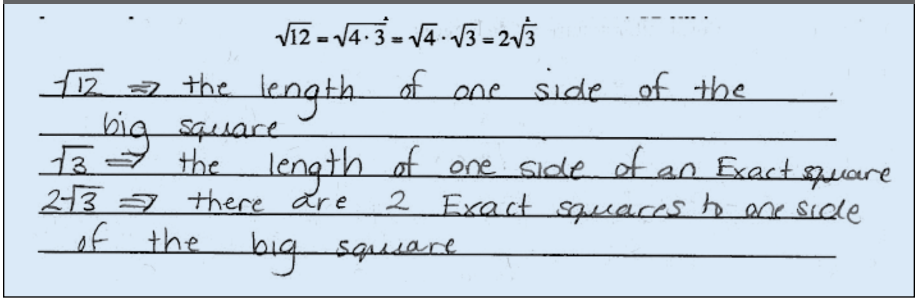
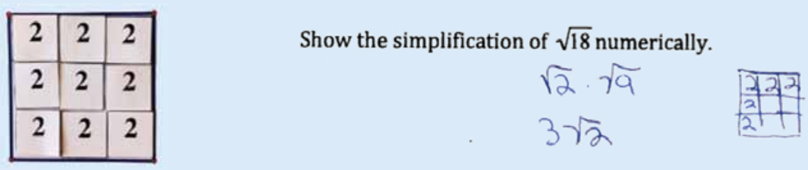


Fig. 6 Various representations of $\sqrt{18}$ help students connect the models to the process of simplifying.



(a) (b)

High School and College Students

The high school and college students began each square root simplification task by first using the traditional algorithm and then making the geometric model fit their calculations. Although this method inhibited the exploration and discovery process that was intended, the students could still make connections between their knowledge of the procedure and the geometric model. For example, one task asked the students to describe the geometric significance of each quantity used in the symbolic process (see **fig. 5**). After using the manipulatives in the geometric context, one high school student used a rough geometric sketch to confirm her numerical work (see **fig. 6**).

Middle School Students

The middle school students, with no prior knowledge of the traditional algorithm, exhibited interesting ways of investigating the geometric model. First, they resorted to guessing or playing with the exact square model as they attempted to fill in the larger square. Specifically, they tried tiles with area 4 or 5 before realizing that four squares with area of 3 cm^2 filled the space of 12 cm^2. Next, they focused on the exact squares that corresponded to the factors of 18 using the large square. At first, they considered "3s since there will be 6 of them," as one student explained, then moved on to try 5s, eventually finding 20 and then 2×9. They then noted that the number of exact squares was always a square number. For example, with $\sqrt{48}$, students started with 8×6:

[Modeling $\sqrt{48}$]

Jack: It should be 8 six times. No, 6 eight times. I'm looking at what makes 48 because that is how it worked with the rest of them.
Olivia: Four?
Jack: It's not 4.
Olivia: You sure? Why isn't it?
Jack: Let's go to 2. Well, does 7 go into it? No, that's 49.
Olivia: No, 7 shouldn't be.
Jack: Let's do 2 then.
Olivia: Oh, it's 3.
Jack: Oh, you want to know why 3 fits? 'Cause there are 16 of them.
Teacher: What do you notice about the amount of exact squares you use? Look throughout the other examples, too.
Jack: Oh, they're all square numbers.
Olivia: Yeah, they're square numbers. That's cool.

These students reasoned with the geometric model; looked for patterns; and, beginning with the $\sqrt{18}$ task, started purposefully looking for factors. This is evident through their initial choice of 6 exact squares of area 3 cm^2 and assertion that exact squares of area 5 cm^2 should not work. This process of finding factors continued and was clearly articulated by one student as the group worked on simplifying $\sqrt{48}$. By the end of the activity, they made connections between the simplification of square roots, the factors of the radicand, and the presence of square numbers.

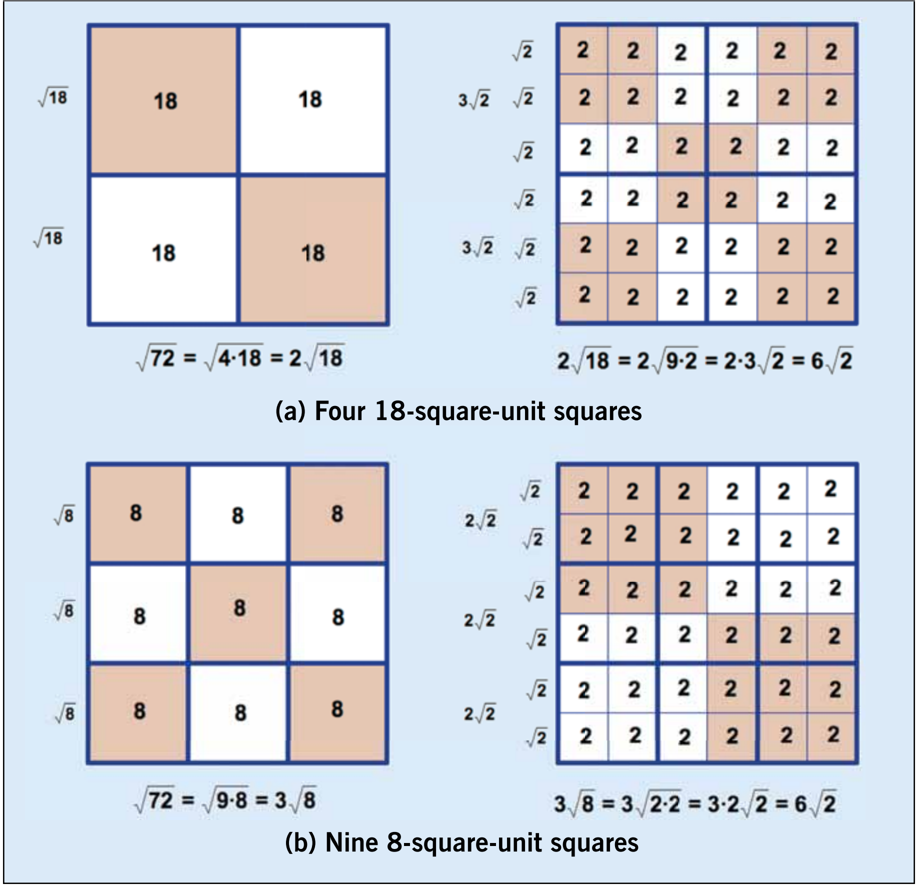
REFINING THE LESSON

Although the initial work with students using this representation is promising, we have more work to do, both in terms of developing activities that allow students to constructively work on more sophisticated radicals and in refining the materials they can use, such as manipulatives.

Manipulatives

One issue we encountered involved the precision of the exact squares manipulatives. Early on, they were created from foam. After we field-tested the foam tiles during work with the preservice teachers, we found the tiles' degree of accuracy to be insufficient. For the next round, the exact squares were laminated. These squares were more accurate than the foam tiles but still contained enough error to provide a false positive (i.e., four 5 cm^2 squares in simplifying $\sqrt{18}$).

Fig. 7 The expression $\sqrt{72}$ can be written a number of different ways. Connecting the exact square representations can help students see why only one way is completely simplified.



The handmade tiles had irrational side lengths, causing a varying degree of inaccuracy in terms of both size and shape. These small or unnoticeable inaccuracies became apparent when multiple tiles were aligned. Although these inaccuracies did not interfere with students' understanding, some were frustrated when implementing the procedure based on the squares. To minimize student confusion and frustration, teachers should be aware of this pitfall when using the tactile exact squares and should identify square roots that might provide false positives. To resolve this issue, we created and are testing a virtual version of exact squares using The Geometer's Sketchpad®.

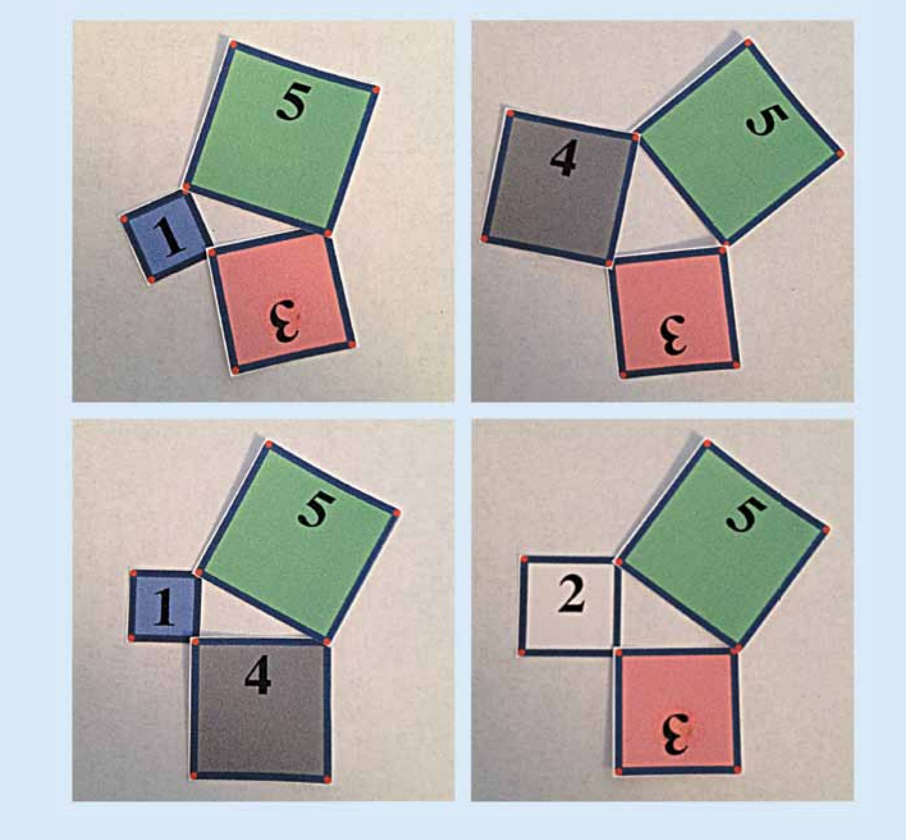
Partially Simplified Radicals

When working numerically, students will often identify a perfect-square factor of the radicand that does not lead to a complete simplification of the radical. For example, if students are simplifying $\sqrt{72}$, they may identify either the factor pairs of 9×8 or 4×18, leading to a partial simplification: either

$$\sqrt{72} = \sqrt{9}\sqrt{8} = 3\sqrt{8} \text{ or}$$
$$\sqrt{72} = \sqrt{4}\sqrt{18} = 2\sqrt{18}.$$

These simplifications can represent an intermediate step to a complete simplification or, if the student does not see further simplification, will mean an end to the process. Our geometric representation, depicted in **figure 7**, illustrates the pictorial relationship between these partial simplifications and could be used to illustrate the numerical connection between them.

Fig. 8 When the areas of the squares on two sides of a triangle sum to the area of the square on the third side, a right triangle is formed.



Other Uses

We are also examining other ways in which squares with whole-number areas could be used to explore mathematical concepts. One idea that may hold promise is using exact squares to demonstrate and explore the converse of the Pythagorean theorem. Although most explorations of the properties of right triangles begin with measuring the side lengths (which are then squared), exact squares allow the student to examine the triangles formed by different combinations of squared side lengths (see **fig. 8**).

A POWERFUL MODEL

Our experiences working with students as they use this geometric model to simplify square roots led us to believe that students are extending their understanding of perfect squares and square roots. Although it was initially used as a model *of* thinking about square roots and how they could be simplified, the individual representation is not an end to the process of simplifying square roots. Rather, the geometric representation can give students a visual connection that can support their use of the traditional numerical method, becoming a model *for* thinking. This visual connection can be continuously accessed as the student thinks through the process needed to simplify square roots, as exemplified by the student work in **figure 6**.

Using multiple representations gives students opportunities to make connections among concepts as well as deepen their understanding of a single concept. This scenario is especially true in classrooms containing special needs, ELL, and gifted students. Currently, the process for simplifying square roots has been limited to one model, the numerical hunt for perfect square factors. The geometric model detailed in this article provides an alternative. The power created by connecting these representations as a path to learning has great potential.

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