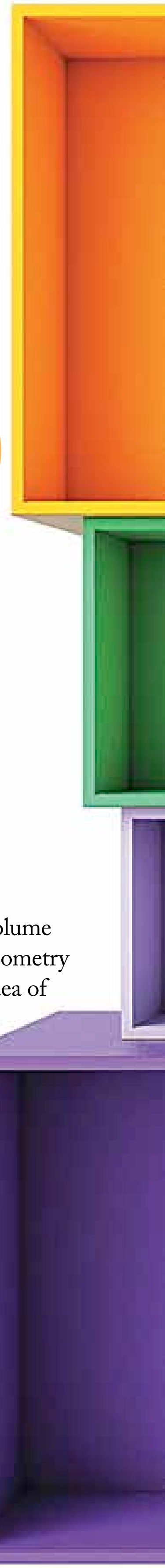
Persevering with Prisms: PRODUCING NETS



A scaling task that incorporates rectangular prisms can net an increase in students' geometric understanding.

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Two mathematics teachers in a small rural school decided to create a task that would engage seventh graders. The goal of the real-world activity was to help students develop geometric and spatial reasoning and to support their understanding of volume of rectangular prisms. The impetus for the task came from the teachers' desire to engage students in mathematical content, in making sense of problems, and in persevering in solving them (CCSSI 2010, p. 7).

The teachers recognized that their students were at various stages in their geometric learning, so they decided to focus on designing a task with multiple entry points that spanned geometric understanding at several grade levels. The areas included measurement from grade 4 (area of rectangles), measurement from grade 5 (interpreting multiplication as scaling, and volume of a right rectangular prism), geometry from grade 6 (continuing the idea of volume of a right rectangular prism and applying the volume formula), and geometry from grade 7 (solving problems including scale, area, volume, and surface area of right rectangular prisms).

The teachers sought to develop a task that had the characteristics of higher-level cognitive demands (Smith and Stein 1998), would provide students

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Fig. 1 Part 1 of the prism task explored a real-world problem.

Your older brother is living on an island in the South Pacific. He lives in a hut and is really hungry for your Mom's homemade chocolate chip cookies. You decide to send him a gift of Mom's cookies. You have a box in the shape of a rectangular prism that is not big enough for all those cookies, so you decide to make your own box with flat pieces of cardboard that Dad brought home from the shop. The box you have is 6 inches long, 4 inches wide, and 3.5 inches tall. The box you want to make needs to be three times taller, three times wider, and three times longer.

PART 1: THE ORIGINAL NET

Directions: Use your colored graph paper to draw a net of the original box, showing the parts that make up the length, width, and height. Each unit on the graph paper should represent 1 inch. Label each rectangle with its dimensions, cut out your net, and fold it. Your grids and labels should be on the outside of the box. Have your teacher check your work before you move to part 2.

Fig. 2 Part 2 of the prism task explored scaling.

PART 2: SCALED NET

Now you are going to make a larger net that has sides that are three times taller, three times wider, and three times longer than your original box, the rectangular prism. Before getting your large sheet of graph paper, make a plan.

1. What will be the dimensions of your new prism?

2. Draw your new net below, showing your new dimensions.

3. How did you determine your new dimensions for the larger box?

4. Use a large sheet of graph paper to draw your new net with dimensions that are three times larger than your original net. Label each rectangle of your net with the new dimensions. After your teacher has checked it, you may cut out this new net and create your new box. Fold your large net together to form a box so that the grid and labels are on the inside of your larger box. Use a few pieces of tape, if needed.

the opportunity to engage in productive struggle (NCTM 2014), and had specific mathematical goals (those things that students could walk away knowing and doing). As the teachers designed the task, they reasoned through their own questions, created models, anticipated outcomes, modified predictions, and began developing a task to engage their students. They considered the words of Driscoll, DiMatteo, Nikula, and Egan that "school children are afforded too few opportunities to locate general results via investigation; rather, the norm tends to be that students are expected to learn and apply results obtained by others" (2007, p. 11). In formulating initial ideas for the task, the teachers shared experiences in their own adult lives in which they needed physical representations to visualize spatial outcomes. These teachers, like so many of their colleagues, had experienced school geometry as the study of "results obtained by others"; therefore, they were eager to help students develop conceptual understanding. They sought to provide an experience that would emulate their visualizations as a way to support students' geometric thinking.

PLANNING THE TASK

The teachers considered their students' mathematical understanding in geometry and recognized the need for students to understand the concept of volume, solve real-world problems involving volume, and construct and describe relationships between geometric figures. They were aware of the Thinking through a Lesson Protocol and wanted to "move beyond the structural components often associated with lesson planning to a deeper consideration of how to advance students' mathematical understanding" (Smith, Bill, and Hughes 2008, p. 137). On the basis of their assessment of student needs, they decided to create a task in which students would have to draw a net and then scale it up so that each dimension would be three times larger than the original rectangular prism. Before working on this task, their students had not created nets and had very limited, if any, experience with constructing three-dimensional objects. The teachers built their task goals around three essential understandings of geometry found in NCTM's Essential Understanding series:

- Decomposing and rearranging provide a geometric way of both seeing that a measurement formula is the right one and seeing why it is the right one. (Essential Understanding 1a)
- Geometric thinking involves developing, attending to, and learning how to work with imagery. (Big Idea 2)

Fig. 3 In part 3, volume represented the culminating focus of the activity.

PART 3: INVESTIGATING VOLUME

In this final part of our task, we are going to investigate the relationships between the smaller prism and the larger prism and consider the volume of each of your prisms.

5. Predict how many of your small boxes will fit inside your larger box.

6. Now test it. How many small boxes will fit on the bottom of your large box?

7. How many layers of small boxes will fit inside your large box?

8. From your test, how many small boxes do you now believe will fit inside your larger box?

9. Your small prism has the dimensions of 4 inches, 6 inches, and 3.5 inches. How many 1-inch cubes could fit inside your smaller prism? Show your work.

10. Your scaled up prism has dimensions of ____ units, ____ units, and ____ units. How many 1-inch cubes could fit inside your larger prism? Show your work.

11. How do you find the volume of any rectangular prism?

12. What is the relationship between the volume of your first box and your second box?

13. Your table partner, Sam, predicted that the volume of your new box would be three times the volume of the original. What could Sam have been thinking when he predicted this, and why was his prediction incorrect?

14. Your other table partner, Jessie, predicted that the volume of your new box would be 27. What could Jessie have been thinking when she predicted this, and why was her prediction incorrect?

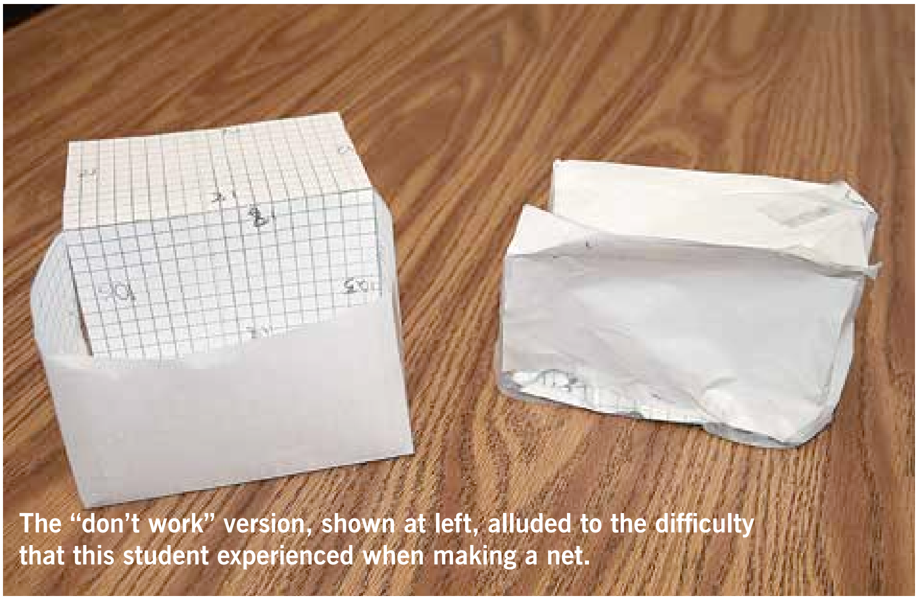
- Geometric awareness develops through practice in visualizing, diagramming, and constructing. (Essential Understanding 2c) (Sinclair, Pimm, and Skelin 2012, pp. 7–8)

After determining the mathematical focus of the task, the teachers created the context. They used the realistic situation in **figure 1**, knowing that students could relate to the experience of having relatives who live in distant locations.

Students were instructed in part 1 to use their sheet of graph paper to draw a net of the original box, label all dimensions, and cut out their nets, making sure that the grid and the labels appeared on the outside of their prisms. Note that the teachers showed students an example of the rectangular prism. In the second part of the task, students were asked to make a scaled model of their original net, with sides that were three times taller, three times wider, and three times longer, using the same sized graph paper. Once they had created their scaled prism, they were instructed to cut it out and tape it together so that the grid was on the inside of the box. The purpose was to have students compare the size of the smaller prism with the larger prism to determine how many smaller prisms would fit inside the larger prism and ultimately to generalize volume of any rectangular prism.

In part 2 of the task (see **fig. 2**), teachers attempted to address student misconceptions that they had anticipated while planning the activity (Smith, Bill, and Hughes 2008). One potential misconception was that students would believe that only three of the smaller prisms would fit inside the larger prism because they scaled the smaller prism's dimensions three times. Questions on the task sheet asked students to first predict how many of the smaller prisms would fit inside the larger prism; then, students were to physically test their thinking. Additionally, before the activity, the teachers used formative assessment strategies and asked students to explain how to find the volume of a rectangular prism. Although students had been previously exposed to the formula for calculating volume, only a small percentage of the students could remember how to find the volume of a rectangular prism. The teachers included a question that asked students to critique the reasoning of another student: "Your table partner, Sam, predicted that the volume of your new box would be three times the volume of the original. What was Sam thinking when he predicted this, and why was his prediction incorrect?"

To further scaffold student thinking about volume and encourage perseverance, teachers also included questions regarding how many smaller boxes would fit on the bottom of the larger box and how many layers of the smaller prism would fit inside the larger prism (see **fig. 3**). Their intent was to help students begin to visualize the volume



The “don’t work” version, shown at left, alluded to the difficulty that this student experienced when making a net.

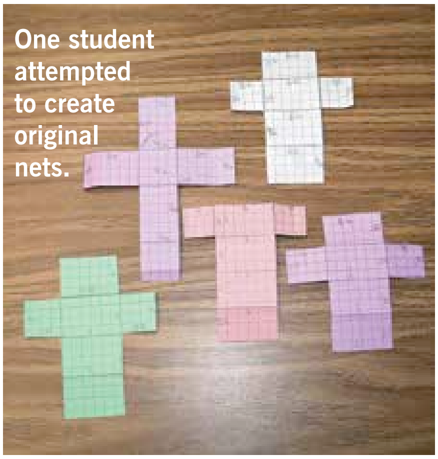
Students often used hand movements to increase understanding.

formula by seeing that three of the smaller prisms would fit exactly along each of the dimensions, therefore generalizing length × width × height. They also wanted to ask questions that would support students in focusing on the mathematical ideas of the task (Smith, Bill, and Hughes 2008).

IMPLEMENTING THE TASK

As teachers introduced the task, they drew a three-dimensional model of the original box on the whiteboard with side lengths labeled 3.5 inches, 4 inches, and 6 inches to help students draw nets and construct prisms. They wanted students to be able to visualize the components of the net coming together to create the prism. They also aspired for students to understand what it would be to see through the object, meaning, to conceptualize the existence of space inside the prism (Sinclair, Pimm, and Skelin 2012). It was challenging for students to construct the geometric figure from a net on graph paper. For example, Kelly kept redrawing her net and saying, “This one doesn’t work.” For some students, including Kelly, the precision required to create the net was difficult. The teachers conjectured that their students lacked prior opportunities to decompose and rearrange figures or to visualize, diagram, and construct, referencing that imagery, an important transitional skill from elementary school to middle school, requires being able to see through or beyond concrete objects (Sinclair, Pimm, and Skelin 2012). The teachers worked to develop among their students, including Kelly, the concept of seeing through objects.

As students created their nets, many of them were physically going through the motions using their hands, showing the front and back, the top and bottom, and the ends of the box as they laid it down in their minds from a three-dimensional box to a flat net. Jacki created five nets for the original task before she was successful and had the correct dimensions. Once students reasoned through the creation of the original net, they seemed to have little difficulty scaling each dimension to create their larger net. However, before moving on, students were asked to predict how many of the smaller prisms would fit inside the larger prism. Some predicted 3, as teachers had anticipated. Only a few predicted 27 and others predicted 9 or 18. Most predicted what seemed like



One student attempted to create original nets.

a random estimate, suggesting a lack of understanding about volume and its connection to changing dimensions. Despite the challenges the students faced with the rigor of the task in comparison with their former experiences, they persevered as they worked to create the larger box for the cookies.

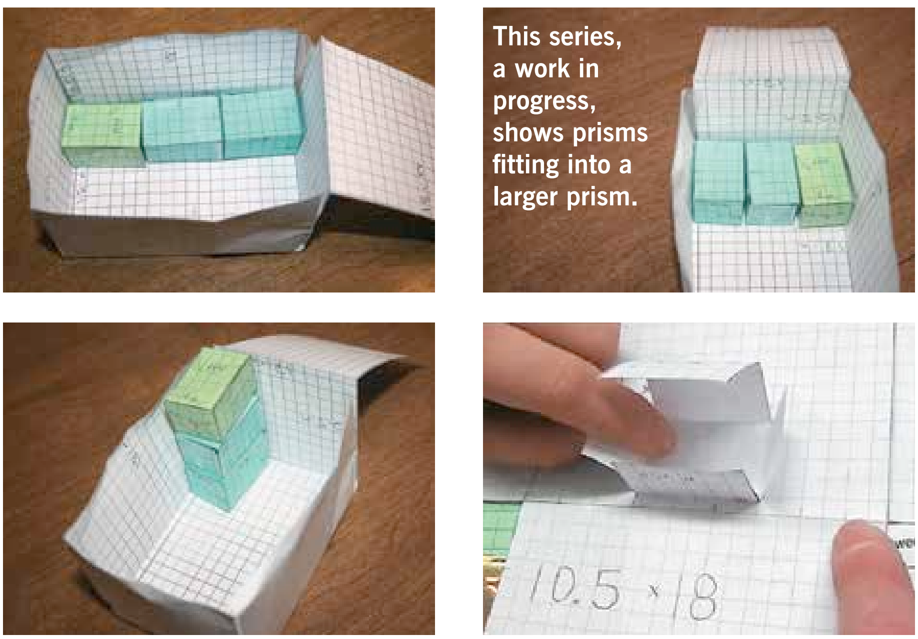
GENERALIZING VOLUME BY DIMENSION

Once students had successfully created their original net and the scaled version, the teachers asked them to decide how many small prisms would fit across each dimension of the larger prism. By physically comparing the smaller prism's side of 4 inches to the larger prism's corresponding side of 12 inches, students could see that three smaller prisms equaled the larger dimension. Using this same process, students discovered that the relationship applied to all dimensions of the larger prisms.

Constructing their knowledge that three of the smaller prisms fit across each dimension of the larger prism triggered numerous aha! moments as the students persevered with solving the problem. One student who had been left to grapple with the generalization of volume had constructed a formula by the time the teacher had returned. When asked how he figured it out, the student said, "I finally figured out that three of these, the width of it is three of these, and then the height is three of these, so now I know that I need to multiply the length times width times height to get the volume."

Another student had a breakthrough moment when he realized that to find volume, he should not just add all the surface areas:

> Well since it's three dimensional, there's a space inside it. . . . [Gasp] Oh, the reason is because since there's an area inside them, all you have to do is multiply this by this [points to base of prism and its height] because then that would give you the volume because it would be like you were just putting a ton of layers over and over.



This series, a work in progress, shows prisms fitting into a larger prism.

This student came to conclusions about the meaning of volume and how to calculate volume by recognizing the ability to see through the prism (Sinclair, Pimm, and Skelin 2012). From this understanding, he generalized the concept of volume as "layers over and over" to make a connection between seeing through the object and calculating volume.

BALANCING EXPLORATION WITH REFLECTION

The teachers were pleased at the rigor of this task, the perseverance of the students, and the extent to which it afforded multiple opportunities for students to engage in geometric habits of mind (Driscoll et al. 2007). Of special interest was balancing exploration with reflection, which is characterized by multiple attempts with intermittent periods of reflection to assess one's progress. This habit of mind is represented by students making and testing attempts, evaluating their productivity, and reflecting on how prior experiments could inform future attempts. Driscoll et al. (2007) found that "one characteristic of successful problem solvers is their metacognitive capacity to balance exploration with taking stock of the productivity of their explorations, then deducing where to take their exploration next" (p. 14). The teachers recognized that many students engaged in a metacognitive process, which led to understanding why prior attempts had been unsuccessful. Some students could see that they had to simply change one dimension, for example, for all sides to meet:

Teacher: How many times did you [make the net]?
Samantha: This was my fourth time.
Teacher: How did you get it right this time?
Samantha: Well, the first time I did it . . . I got it right and everything [the sides fit together], but then I realized that I did it wrong.
Teacher: In what way did you do it wrong?
Samantha: On this one I did four and a half, four inches and then three point five inches. On the second try, I did all of them six by three and a half and six by three and a half, and six by three and a half.
Teacher: So you didn't have that four-inch dimension?
Samantha: Yeah.

Most students balanced exploration with thoughtful reflection and continued to persevere as they made more attempts to create their first successful net of the original prism.

REFLECTIONS ON PRODUCTIVE STRUGGLE

Like their students who balance exploration with reflection, effective teachers reflect as their students engage with tasks. They are continually exploring to refine their practices to plan lessons that are designed around how and what "students are to learn and understand from their work on the task, not just what they will do" (Smith, Bill, and Hughes 2008, p. 135). Through productive struggle and a focus on balancing exploration with reflection, the teachers in this school provided a task that encouraged students to explore volume while considering their own thinking and engaged students in many of the Standards for Mathematical Practice (CCSSI 2010).

Because the task was designed with multiple entry points, students were encouraged to persevere and experience success at all levels of completion. Some were only able to successfully create the first net, requiring multiple days for this component of the task; once their first net was completed, however, these students felt triumphant. Others required only two or three attempts until they successfully completed their first net and then easily scaled their original to dimensions that were three times greater. A few students reached the point of grappling with volume relationships, with some ultimately generalizing the volume formula. The teachers were pleased with the growth they had seen among their students, both with content understanding and their ability to persevere. They recognized that providing opportunities for productive struggle created opportunities for learning that supported mathematical understanding and development that otherwise may not have occurred.

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BIBLIOGRAPHY

Common Core State Standards Initiative (CCSSI). 2010. Common Core State Standards for Mathematics. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. http://www.corestandards.org/wp-content/uploads/Math_Standards.pdf

Driscoll, Mark, Rachel Wing DiMatteo, Johannah Nikula, and Michael Egan. 2007. *Fostering Geometric Thinking: A*

Guide for Teachers, Grades 5–10. Portsmouth, NH: Heinemann.

Hiebert, James, Thomas Carpenter, Elizabeth Fennema, Karen Fuson, Diana Wearne, Hanlie Murray, Alwyn Olivier, and Piet Human. 1997. *Making Sense: Teaching and Learning Mathematics with Understanding.* Portsmouth, NH: Heinemann.

National Council of Teachers of Mathematics (NCTM). 2014. *Principles to Actions: Ensuring Mathematical Success for All.* Reston, VA: NCTM.

Sinclair, Nathalie, David Pimm, and Melanie Skelin. 2012. *Developing Essential Understanding of Geometry for Teaching Mathematics in Grades 6–8.* Reston, VA: National Council of Teachers of Mathematics.

Smith, Margaret, and Mary Kay Stein. 1998. "Selecting and Creating Mathematical Tasks: From Research to Practice." *Mathematics Teaching in the Middle School* (February): 344–50.

Smith, Margaret, Victoria Bill, and Elizabeth Hughes. 2008. "Thinking through a Lesson: Successfully Implementing High-Level Tasks." *Mathematics Teaching in the Middle School* (October): 132–38.

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6.G.2
7.G.3



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