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Slicing a Cube

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Another source of activities can be found in NCTM's *Using Activities from the "Mathematics Teacher" to Support "Principles and Standards,"* edited by Kimberley Girard and Margaret Aukshun (order number 12746; $34.95), which also includes a grid to help teachers choose the activities that best meet the needs of their students.

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As mathematics educators, we know that mathematical maturity and good habits of mind need to be developed and encouraged continually. For students to form these habits, they need opportunities to think and explore mathematically. The activity "Slicing a Cube," in which students are asked to decide whether or not certain figures may be obtained when a plane slices through a cube, provides such an opportunity.

Initially, students are given only a list of geometric figures and drawings of cubes. They will try to create each figure from the list by drawing a plane as it slices through a cube. Later, they will be given transparent cubes. To illustrate the slicing plane, students may fit rubber bands around the cubes or partially fill the cubes with water. The three-dimensional manipulatives will enhance the exploration and mathematical discussion.

This activity allows for a wide range of mathematical questions and arguments—both geometric and algebraic, intuitive and rigorous. It also encourages mathematical discussion and communication among students. Finally, it is fun and versatile. I vary this activity according to the audience, whether high school and college students or middle school and high school teachers. For high school students, I suggest using at least two class periods (for a minimum total time of 90 minutes) to give students time to explore and discuss their ideas.

MATERIALS

Use clear plastic cubes that have one face acting as a lid. After pouring water into a cube, seal the lid so that the cube may be moved and tilted without leaking. Have enough cubes so that students can work in small groups (at least one cube per group). Students will pour water into and out of these cubes. I fill a one-gallon jug (such as a plastic milk container) with water colored with food dye and give the students another container with a wide top to pour the colored water from the cubes. Other materials include a large roll of paper towels and plenty of rubber bands that can fit tightly around the outside of the cubes.

TEACHER'S GUIDE

Students engage more and communicate better when they work in small groups. Hand out **sheet 1** to each student. Usually, I immediately hand out **sheet 2**, which is filled with drawings of cubes. If you have a very advanced class or see that your students are making progress, you may want to wait until they get stuck before handing out this second sheet. Students should label and draw on **sheet 2**, as accurately as possible, those figures that they can obtain by slicing. They should also try to draw some of the figures that they are initially unsure about. This exercise will help them visualize cross sections more clearly. Occasionally, students find the drawing efforts more helpful than working with the three-dimensional cubes. Give students plenty of time (20 or 25 minutes) to discuss, reach conclusions, and draw. I also take a poll and record the results on the board (for each figure, I record the number of "yes," "no," and "not sure" responses). Taking the poll allows students to share their group's results with the rest of the class, generally sparks some discussion, and provides students with a sense of collective achievement as well as some perspective on their progress.

Next, I hand out the cubes and the rubber bands. I place the colored water and the other container in a designated space where students can pour water into and out of their cubes. They will start exploring, refining their answers, and making better drawings. Give them 30 to 40 minutes to explore and finish drawing (this process may take you into your next class). Then you may revisit the poll and record the results again. There should be

Fig. 1 The trouble with rubber bands

more agreement and fewer "not sure" entries. A warning: The surface of the water inside the cubes is always planar, but the rubber bands are a different story (see **fig. 1**). Students should discover that they need to make their "slices" planar if they are using rubber bands. If not, you will need to guide them to this realization.

Devote the remainder of the time to discussing and analyzing the results as well as other issues that arose during the activity. Here are some questions to pick and choose from, or you may devise your own:

1. Which figures from the list can be obtained by slicing a cube with a plane? Which cannot? Why?

2. Must the intersection of a plane and a cube always be a polygon?

3. If we did not succeed in obtaining a figure, does that prove that the figure cannot be obtained?

4. If we did succeed in creating a figure that looks like an equilateral triangle or a regular hexagon, how do we know for sure that the figure can be obtained? What if we were off by very little, so little that our eyes cannot discern the difference?

5. If we begin with a fixed amount of water in the cube, which figures can we obtain? Does the answer change if we start with a different amount of water? How?

6. If we begin with a certain type of polygon (for example, a triangle), what amounts of water can produce it? How does the answer change if we start with a different type of polygon?

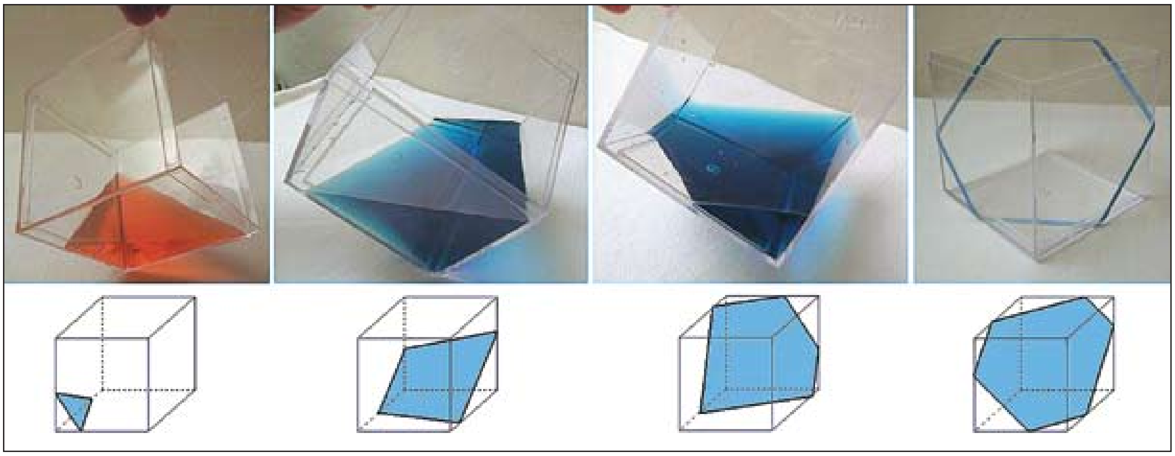

Fig. 2 The polygon can have three, four, five, or six sides.

DISCUSSION

To begin, the intersection of a plane and a cube does not have to be a polygon. The intersection could be empty (as when the plane and cube do not meet at all); it could be a single point (as when the plane meets only a vertex of the cube); or it could be a line segment (as when the plane meets only an edge of the cube). In these cases, the plane does not meet the interior of the cube. Another case in which the plane does not intersect the cube's interior does produce a polygon (a square)—namely, when one face of the cube lies on the plane. We will ignore these "degenerate" cases and assume, from now on, that the slicing plane meets the interior of the cube.

Given that assumption, we know that the intersection must be a polygon. Whenever the plane meets a face of the cube, the result is a line segment on that face. (Note that it is possible for the plane to meet two faces along their common edge, but this possibility changes nothing.) The intersection, of course, is planar, because it is a subset of the slicing plane. So it is a polygon and, moreover, a convex polygon. Recall that a convex set contains the entire line segment joining any two of its points. Clearly, both the cube (viewed as a solid figure) and the plane are convex. From the definitions of *convex set* and *intersection of sets*, any intersection of convex sets is convex.

How many sides can the polygon have? Because the plane contains a point in the interior of the cube, the polygon must meet at least three faces. So the polygon can have as few as three sides. Because the cube has only six faces, the polygon can have at most six sides. So it must be a triangle, quadrilateral, pentagon, or hexagon (see **fig. 2**).

A little informal geometry, set theory, and simple counting rule out the last three figures on the list on **sheet 1**. Ruling out other figures seems harder, and students are likely to get stuck at this point. So let us now consider questions 3 and 4. Later, a geometric observation and a simple algebraic argument (an application of the Pythagorean theorem) will rule out more figures.

Students should realize that failure to produce a figure from the list (with water or rubber bands) may be valuable in conjecturing that the figure is impossible but is not a mathematical proof. Only a mathematical argument can resolve the issue. Similarly, it might seem as if some figures from the **sheet 1** list, such as the equilateral triangle or the regular hexagon, can be obtained. But we need a clear description of how to slice the cube to make sure that the resulting figure is in fact what we claim and not the very close approximation.

I will omit descriptions of the possible figures, but I will give two examples, because these descriptions are a good exercise for students, especially in developing their mathematical communication skills. Depending on the class's grade level or sophistication, it may be advisable for students simply to give descriptions using clear mathematical language, or you may want them to prove that the resulting figures have the required properties that define them. For example, they may use the Pythagorean theorem (three times) to prove that a certain triangle is scalene, or they may use facts about parallel planes and lines to prove that a figure is a parallelogram.

The easiest of all figures is the square, which is obtained when the slicing plane is parallel to a face of the cube. A slightly more challenging description is that of the equilateral triangle. Choose a positive length d that does not exceed the side length of the cube and then choose a vertex of the cube. On each of the three edges that share this vertex, mark the point that is a distance d from it. A plane slicing the cube through these three marked points will produce an equilateral triangle.

Now I want to focus on finding the remaining impossible figures. One way to do this (as we did earlier) is to find some property that a figure must possess if it results from slicing the cube with a plane but that is not shared by some figure from the

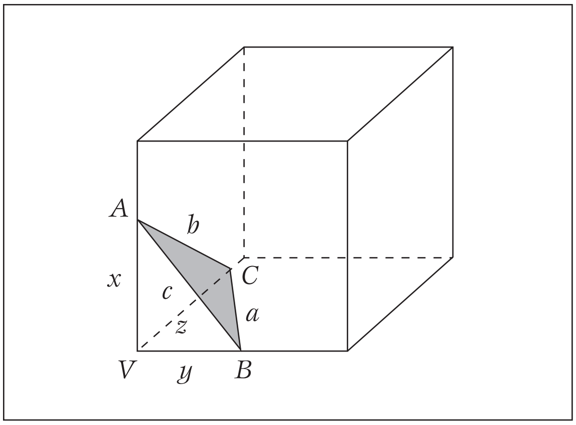

Fig. 3 A right triangle cannot be obtained.

activity sheet list. Two such properties are that (1) all possible figures must be convex polygons and (2) they can have no more than six sides. Another approach is to look at a particular figure from the list and somehow prove that it cannot be obtained. We will do both, in the first case using a geometric property and in the second case using a little algebra.

The geometric observation mentioned earlier is based on the fact that the faces of the cube occur as three pairs of (opposite) parallel faces. If a given plane intersects two other planes that are parallel, the resulting lines are parallel. If a plane slices a cube to form a quadrilateral, then at least two of its sides must be parallel. Similarly, if the result is a pentagon, then it must have two pairs of parallel sides. And if the result is a hexagon, it must have three pairs of parallel sides. Because a kite and a regular pentagon have no parallel sides, these figures are impossible to obtain. Students can be guided toward this realization through such questions as whether all their quadrilaterals have parallel sides or whether they can obtain one without any parallel sides. Similar questions about pentagons and hexagons might lead to more discoveries.

One last figure from the list that cannot result from slicing a cube with a plane is the right triangle. Here is a proof by contradiction. Suppose that a plane could slice a cube and form a right triangle ABC with its right angle at C. Let V be the vertex of the cube that is common to the three faces that meet the slicing plane to form the triangle (see **fig. 3**). Let a, b, and c be the lengths of the sides of $\triangle ABC$ opposite A, B, and C, respectively, and let $x = VA$, $y = VB$, and $z = VC$. Note that AVB, AVC, and BVC are also right triangles, with right angles at V. By the Pythagorean theorem,

$$c^2 = a^2 + b^2 = (y^2 + z^2) + (x^2 + z^2)$$
$$= (x^2 + y^2) + 2z^2 = c^2 + 2z^2.$$

Subtracting c^2 from the first and last expressions in this string of equalities, we see that $z = 0$. But then $V = C$ and the slicing plane contains a face of the cube, so that a triangle cannot be formed at all. This is a contradiction, and therefore a right triangle cannot be obtained. Giving students the setup of **figure 3** and asking them to look for right triangles, write down equations relating the sides of these triangles, and make substitutions as necessary may be enough to allow them to reach this conclusion.

The remaining ten figures from the list can all be obtained, and, as mentioned earlier, it would be good for students to describe precisely how. But the rest of the discussion here will focus on questions 5 and 6. It turns out that the figures we can obtain with the water depend on the amount of water in the cube. So which figures can we obtain with what amount of water? The only feature I will consider is the number of sides of the figures. But, first, a pedagogical remark: Students do not always realize that with a fixed amount of water in their cubes, they limit the possible cross sections formed by the water surface, especially when each group has only one cube and there is not much interaction between groups. Students should be guided to vary the amount of water in their cubes. Asking them to try to obtain a triangle with their cubes filled halfway might provoke reflection.

This activity allows for mathematical questions and arguments both geometric and algebraic, intuitive and rigorous

Now let us answer questions 5 and 6. First, any figure that can be obtained by slicing a cube with a plane can be produced with some amount of water: Simply tilt the cube so that the slicing plane is horizontal and fill the cube to the required level. A more interesting observation is that, in fact, we never need more water than an amount equal to half the volume of the cube. One way to see this is to switch the role of the water with that of the air in the cube and the floor with the ceiling. A little more formally, let us define

$$p = \frac{\text{volume of water in the cube}}{\text{volume of the cube}}.$$

Then the figures obtained with p and with $1 - p$ are the same, and we need to consider only $0 < p \leq 1/2$.

A triangle may be obtained for $0 < p \leq 1/6$. At $p = 1/6$, we can form the equilateral triangle shown in **figure 4**. Students may deduce that this triangle corresponds to $p = 1/6$ by computing the volume of the resulting pyramid or noting that six such pyramids fill the cube. Note that to form a triangle, pentagon,

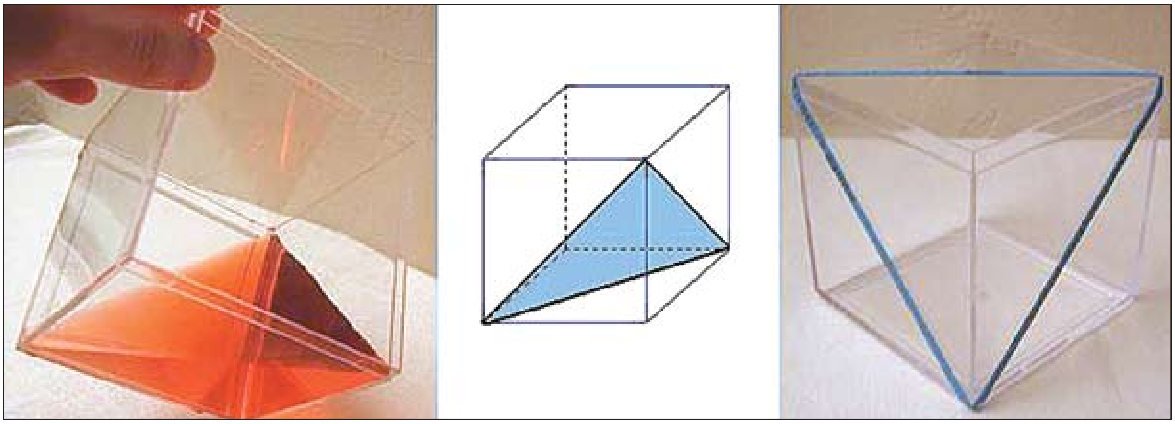

Fig. 4 Equilateral triangle at $p = 1/6$

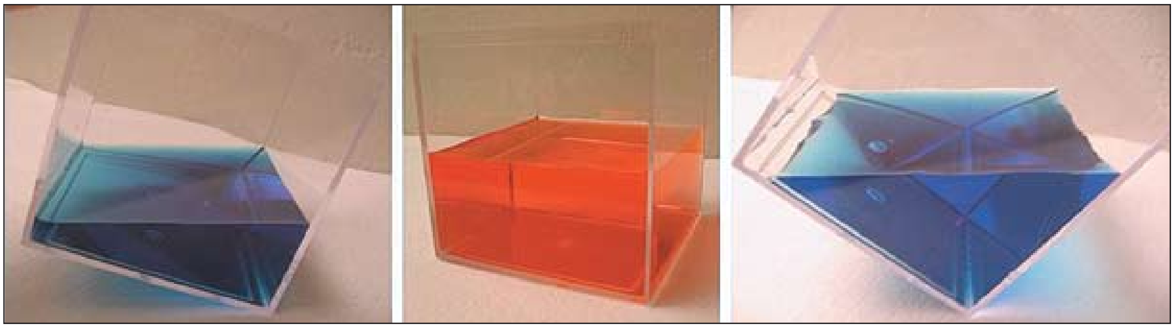

Fig. 5 When an edge of the cube is parallel to the floor, only rectangles can be obtained.

or hexagon, the cube must be tilted in such a way that there is a unique lowest vertex. Otherwise, if an edge is parallel to the floor, only rectangles may be formed (see **fig. 5**). For $1/6 < p \leq 1/2$, the water level will inevitably go higher than at least one of the vertices adjacent to the lowest vertex, and the resulting figure will have more than three sides.

A quadrilateral may be obtained for all values, $0 < p \leq 1/2$. Obviously, you can always get a square (you can also always get a nonsquare rectangle or rhombus). The pentagon is harder. It may be obtained for $0 < p \leq 1/2$ but not for $p = 1/2$. When $p = 1/2$, no matter how we tilt the cube, the surface of the water passes through its center, and the resulting symmetry rules out a pentagon. Finally, a hexagon may be obtained for $1/6 < p \leq 1/2$. This case is sort of a complement to the triangle case. Below $p = 1/6$, the water level cannot "catch" all three upper faces (i.e., those that share the highest vertex). **Table 1** summarizes which type of polygon may be produced with what amount of water.

Table 1

Polygons That Can Be Obtained with Different Amounts of Water (p = volume of water in the cube or volume of the cube)

	$0 < p \leq 1/6$	$1/6 < p \leq 1/2$	$p = 1/2$
Triangle	Yes	No	No
Quadrilateral	Yes	Yes	Yes
Pentagon	Yes	Yes	No
Hexagon	No	Yes	Yes

CONCLUSION

The value of this activity lies not in the importance of its topic. Its value lies in using mathematical knowledge that students already have to reason and explore mathematically in an unfamiliar situation. When students engage in problem-solving activities like this one, they not only improve their problem-solving skills but also deepen and better retain the mathematics that they bring to their solution attempts. As NCTM's *Principles and Standards for School Mathematics* argues, "Solving problems is not only a goal of learning mathematics but also a major means of doing so" (2000, p. 52). NCTM also advocates that students make connections between mathematical ideas to deepen their understanding and their view of mathematics as a coherent subject. This activity can help them bridge the informal and intuitive with the formal and rigorous, the visual and geometric with the algebraic, the particular and concrete with the general and abstract. These are complementary, not conflicting, ways of thinking. Students who learn to appreciate the benefits of their interplay will be able to enjoy the beauty of mathematics and mathematical thinking. Finally, as students try to find answers and then explain them, they enhance their oral and written communication skills. This step, important in itself, also helps solidify students' understanding.

Author's note: I first conducted this activity in summer 2002, while serving as a leader of a geometry professional development institute. The colleague who brought this activity to my attention during the planning stage does not recall its origin, and I have been unable to trace it. Over the years, I have added substantially to the first version I used, including some impossible figures and questions 2, 5, and 6.

Cubes of side length 4 inches, a very convenient size, are available from TAP Plastics (www.tapplastics.com; see under Plastic Containers); similar cubes are available from AMAC (www.amacbox.com) or Melmat (www.melmat.com). They cost about $4.00 each. For an online manipulative to complement this activity, go to the National Library of Virtual Manipulatives (http://nlvm.usu.edu). This Web site contains a wealth of useful manipulatives to enrich lessons or activities. Click on Virtual Library; in the matrix, select Geometry and Grades 6–8; and then go to Platonic Solids–Slicing. Students will be able to slice the cube and the other platonic solids.

Members who wish to use an MS Word version of this month's activity sheets in a classroom setting can download the file from NCTM's Web site, www.nctm.org. Follow links to *Mathematics Teacher* and choose this issue. From the Departments, select Activities and look for the link to the activity sheet.

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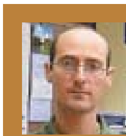

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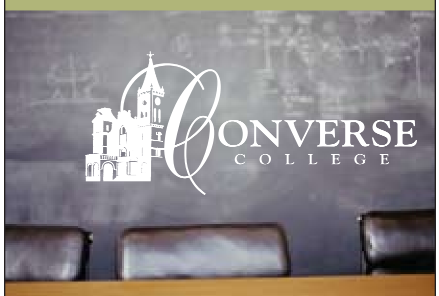


Slicing a Cube

Sheet 1

If a plane slices through a cube, which of the following figures may be obtained? Write "yes," "no," or "not sure" to the left of each figure:

_____ 1. Equilateral triangle

___ 2. Isosceles triangle (not equilateral)

_____ 3. Scalene triangle

___ 4. Right triangle

_____ 5. Square

___ 6. Rectangle (not square)

_____ 7. Rhombus (not square)

___ 8. Kite (not a rhombus)

_____ 9. Trapezoid

____ 10. Regular pentagon

_____ 11. Pentagon (not regular)

___ 12. Regular hexagon

_____ 13. Hexagon (not regular)

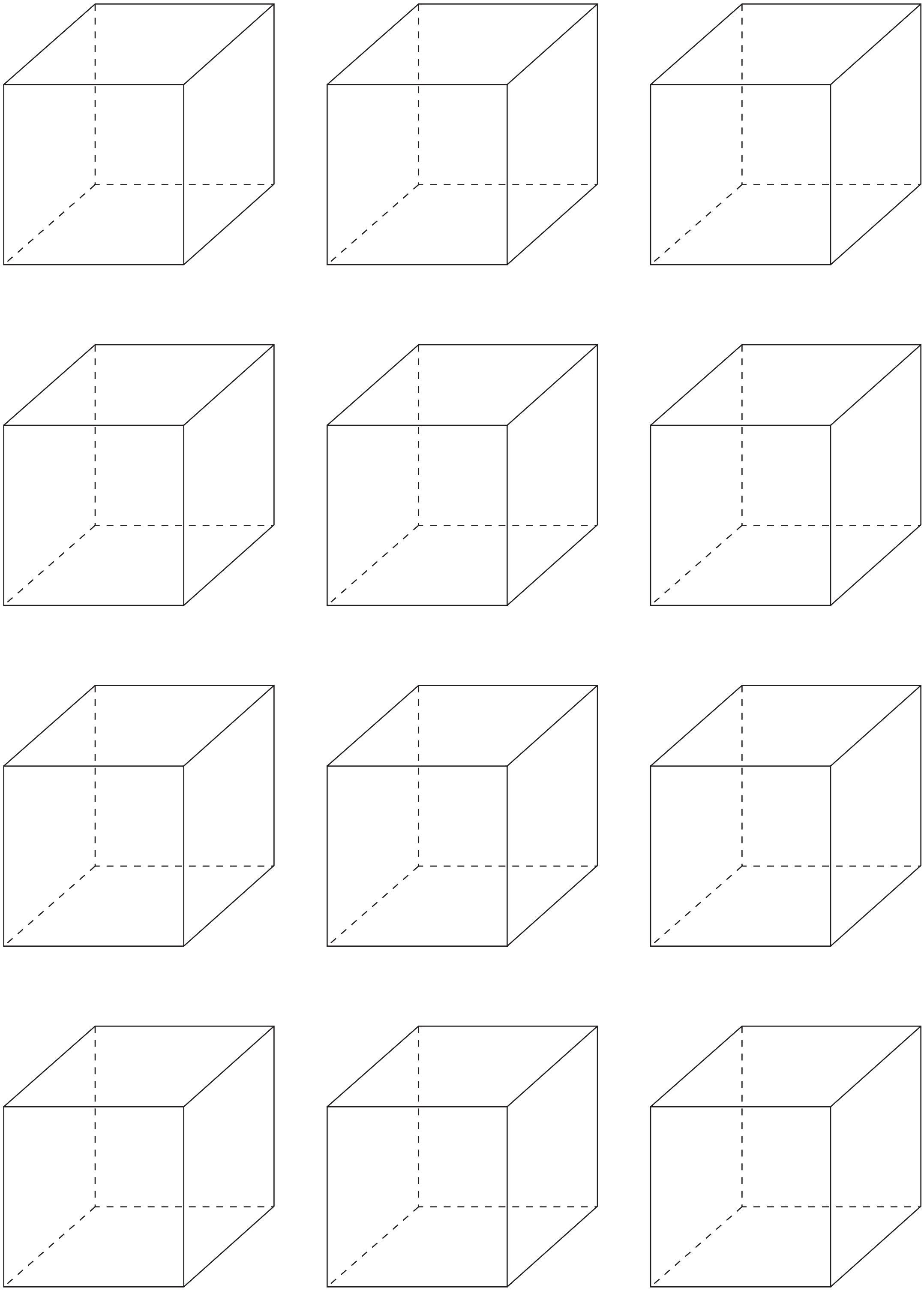
___ 14. Heptagon

_____ 15. Polygon with more than 6 sides

____ 16. Nonconvex polygon

Slicing a Cube

Sheet 2



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