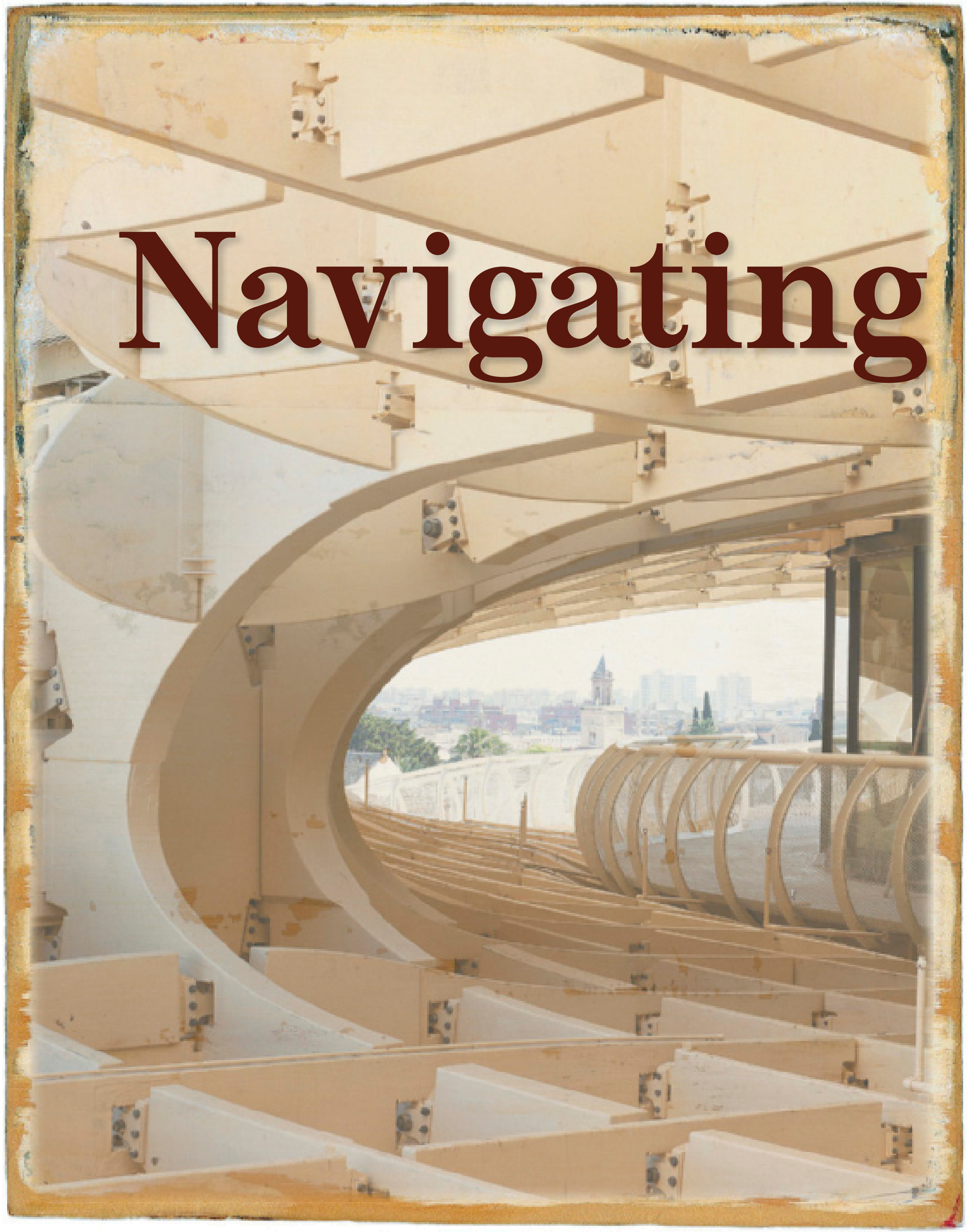
Navigating



DAVID FRANCK

between the Dimensions

Two classroom activities—the Flatland game and sliceforms—are useful vehicles for student exploration of the geometric interplay between the dimensions.

Julian F. Fleron and Volker Ecke

Yet I exist in the hope that these memoirs, in some manner, I know not how, may find their way to the minds of humanity in Some Dimensions, and may stir up a race of rebels who shall refuse to be confined to limited Dimensionality.—A. Square, *Flatland*

Generations have been inspired by Edwin A. Abbott's profound tour of the dimensions in his novella *Flatland: A Romance of Many Dimensions* (1884). This well-known satire is the story of a flat land inhabited by geometric shapes trying to navigate the subtleties of their geometric, social, and political positions. (A new animated version of this classic is now available as well; see Johnson and Travis [2007].) In this article, we introduce the Flatland game and a related project, which inspire repeated trips between Flatland and our three-dimensional world. Both activities are effective vehicles for students' exploration of the geometric interplay between the dimensions.

As inhabitants of a three-dimensional world, we authors feel that solid geometry could play a more prominent role in secondary school curricula and that students' experiences could move beyond explorations of spheres, cones, cylinders, prisms, pyramids, and the platonic solids. The activities introduced here allow students to explore our three-dimensional world in its full diversity, using any solid object as the focus of each activity. This liberation, we believe, honors the spirit of A. Square of Flatland, who urged humans to work to have our minds "opened to higher views of things" (Abbott 1884, p. 4). In more concrete terms, the activities provide rich opportunities to support NCTM's Geometry Standards (NCTM 2000) (see **fig. 1**).

CONNECTIONS TO NCTM'S GEOMETRY STANDARDS

Among the many connections, our activities help students "use visualization, spatial reasoning, and geometric modeling to solve problems," addressing the following grades 9–12 expectations:

- "Draw and construct representations of two- and three-dimensional geometric objects using a variety of tools"
- "Visualize three-dimensional objects from different perspectives and analyze their cross sections"
- "Use geometric models to gain insights into, and answer questions in, other areas of mathematics"
- "Use geometric ideas to solve problems in, and gain insights into, other disciplines and other areas of interest such as art and architecture"

Fig. 1 The activities presented in this article have many connections to NCTM's Geometry Standards (NCTM 2000, p. 308).

REAL-LIFE APPLICATIONS

Humans live in the three-dimensional world of Spaceland, yet many of our means of investigation, visualization, and construction are two dimensional—images captured on paper and computer screens. Every day we spend a great deal of time trying to navigate between two and higher dimensions, playing the roles of radiographers and builders described below.

To set the context for our activities and to help pique student interest, we briefly provide real-life applications of the interplay between the dimensions. Some of these applications will be familiar to you and perhaps to your students; others may not be. Space prohibits us from describing these applications in detail, but we have compiled information, short activities, and additional links related to each area online at http://www.westfield.ma.edu/ecke/flatland. We encourage you to use this material in your lessons because it helps motivate students and enhance their understanding.

Real-life applications that require the ability to navigate the dimensional cycle between two- and three-dimensional spaces successfully include these:

- Mapping—Topographic maps, topographical profiles, and contour lines; a host of connections with modern geographic information including a GPS and Google Earth™
- Medical imaging—X-rays, CAT scans, and many other forms of medical applications
- Design—Architecture and architectural design; blueprints; computer-aided design (CAD) in product development, engineering, and woodworking; general three-dimensional modeling software such as Google SketchUp™
- Building—Construction trades, landscapers, and regional planners; work from blueprints
- Three-dimensional scanning, three-dimensional printing, and stereolithography
- Holography, IMAX, and 3D movies
- Three-dimensional simulation and virtual reality CAVEs used in surgical training and flight training
- Digital animation and computer gaming

These activities are related to a wealth of real-life applications. We have used these activities in a variety of settings, including with high school and introductory college-level audiences as well as in teacher professional development, with much success. Obviously, there is a great deal here both to inspire and to help prepare the next generation of students for the mathematics they will need as architects, tradespeople, engineers, artists, builders, graphic designers, medical specialists, animators, and teachers.

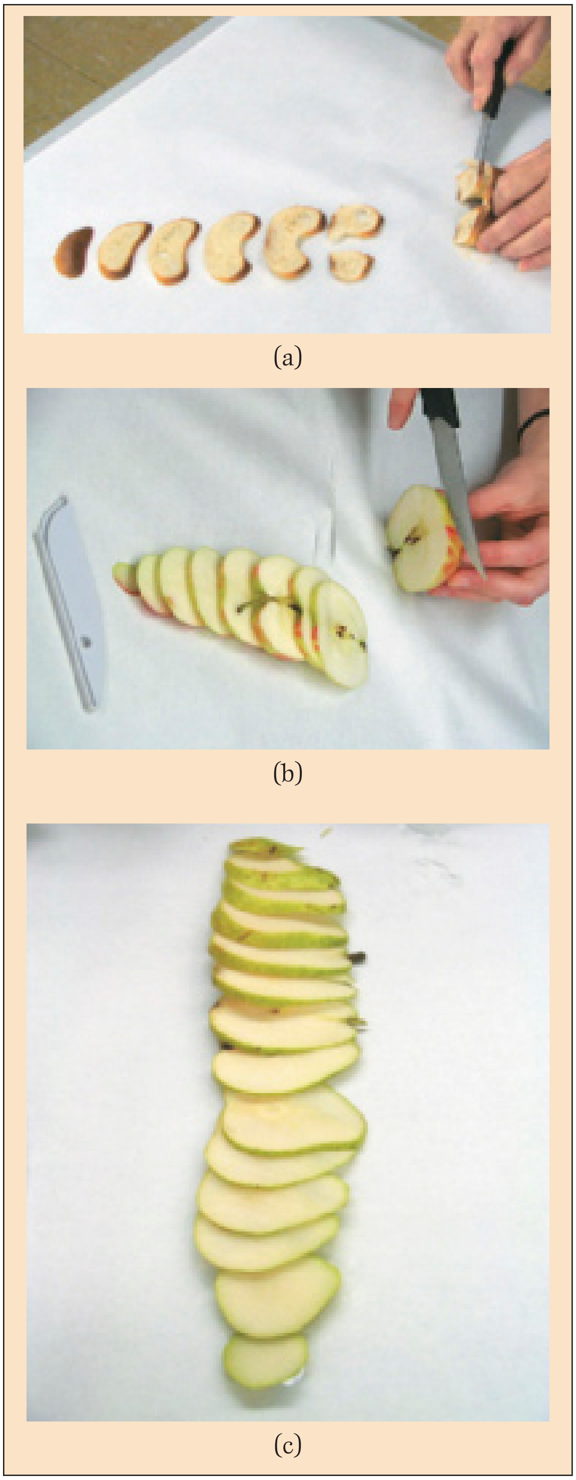
FLATLAND AND SLICING

The interplay between the dimensions can be investigated in many ways. Robbins (2006) argues strongly on behalf of projection, and Frantz and colleagues (2006) show how perspective drawing can be effectively explored in mathematics classrooms—both valuable approaches. Here we will focus on the slicing technique popularized in *Flatland*.

Teachers can introduce slicing and the mathematics of cross sections in many ways, but the most direct is simply to have students physically slice real objects into cross sections. We suggest lining tables with large sheets of paper or disposable tablecloths for easier cleanup. Denser and drier objects seem to work better; apples, zucchini, bagels, sponge cake, Play-Doh, cheese, and pineapple all work well. Students can use either sharp knives or thin wire to cut the cross sections (see **fig. 2**).

Once students have sliced the objects, their natural impulse is to arrange their cross sections linearly, showing the solid object as a sequence of cross-sectional slices. If the same object is sliced in a different direction, students will discover the dramatic impact that this change in orientation has on the shape of the cross sections. Encourage students to draw pictures of the sequences of cross sections in their notebooks.

This activity provides a developmentally important experience in understanding the interplay between the dimensions. However, as an analytical tool, physically slicing a few routine objects is only a beginning. We need to expand this experience to encompass a more diverse collection of geometric objects and activities to help students to develop a deeper, more sophisticated ability to navigate the dimensional cycle shown in **figure 3**.



JULIAN F. FLERON

Fig. 2 Students can use knives to cut cross sections of a bagel (**a**), an apple (**b**), and a pear (**c**).

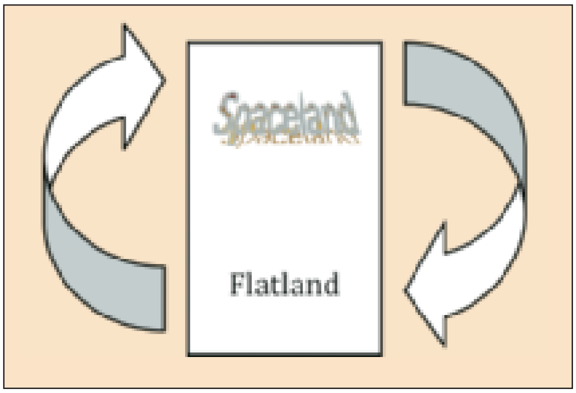


Fig. 3 The dimensional cycle relates two and three dimensions.

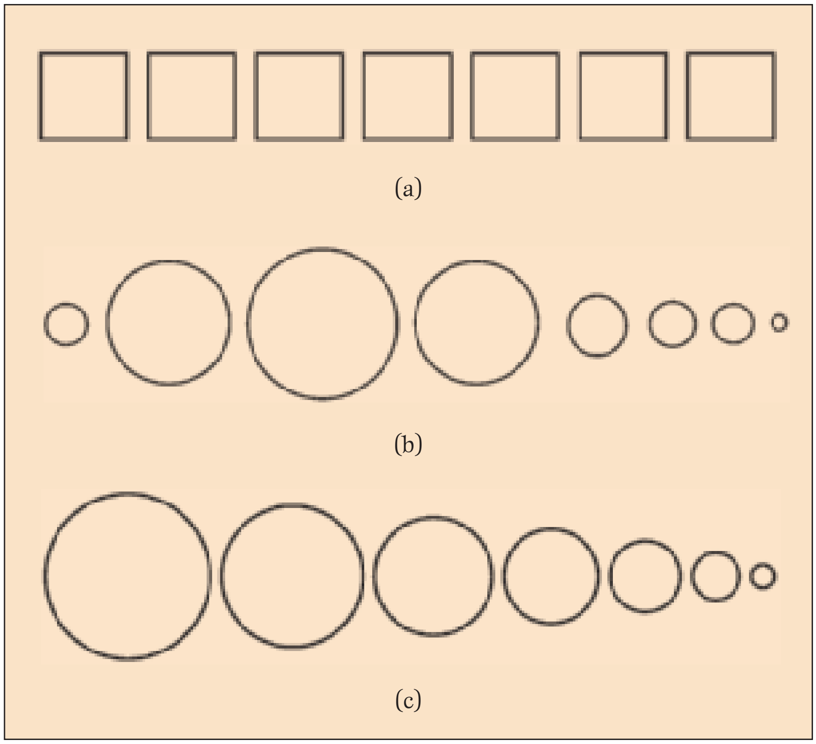


Fig. 4 Which solid objects produce these slices?

THE FLATLAND GAME

Having toured the higher dimensions, our hero A. Square knows why Sphere appeared to him as circles of continuously changing size: They were the cross sections of Sphere. In the Flatland game, we put ourselves in A. Square's position. Our task is to determine the identity of a solid object passing through Flatland from a series of parallel cross sections. Three examples are given in **figure 4**. Can you guess what three secret solids make these cross sections as they pass through Flatland? (Answers are provided at the end of the article.)

Determining the solid is the inverse problem of determining the cross sections. If we combine determining the solid with the direct problem of slicing, we have a way to complete the dimensional cycle in **figure 3**. This is a developmentally important activity that is easily made dynamic by turning it into a game—the Flatland game.

In the Flatland game, a team of radiographers chooses a solid object and draws a sequence of cross-sectional slices, one at a time. The radiographers compete against teams of builders who attempt to determine the identity of the solid object from the cross-sectional clues.

Rules and Roles for the Flatland Game

Instructions for the Flatland game follow:

1. Choose one or more radiographers. The radiographer can be a single person or a small team that has chosen an illustrator.
2. Choose teams of builders. Teams can consist of a single person, if necessary; it is best if a few small teams compete.
3. The game starts with the radiographers determining among themselves a solid object from Spaceland; the identity of this solid will be the focus of the game.
4. The illustrator for the radiographers begins play by drawing a single cross section, as viewed from above, of the secret solid.
5. The builders attempt to guess the identity of the secret solid.
6. The illustrator for the radiographers then draws another cross section of the secret solid. This cross section must be parallel to earlier ones and must be revealed consecutively as they would be if the object actually passed through Flatland.
7. Steps 5 and 6 are repeated until—
   (*a*) a team of builders correctly guesses the identity of the secret solid, in which case this team is declared the winner, or
   (*b*) there are no more cross sections to draw and the radiographers are declared the winner.

This game encourages multiple circuits around the dimensional cycle. Each time, students draw, visualize, construct, slice, analyze, and reconstruct—in short, they develop critical tools for navigating between the dimensions. No longer are their investigations of solid objects limited; chairs, coffee cups, and people are the fodder for their geometric investigation. The interplay between cross sections and real-life solids provides a powerful opportunity for reflection on the nature of the three-dimensional world we inhabit.

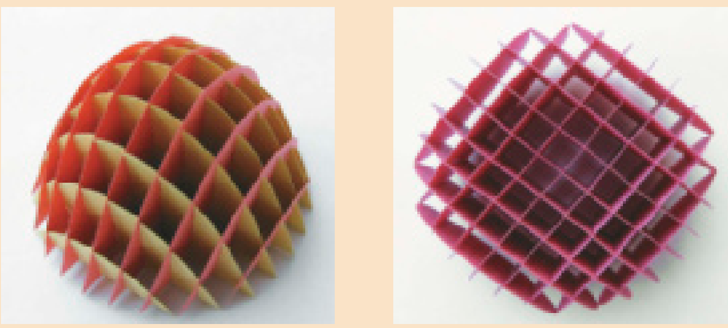


Fig. 5 Sliceforms may include a paraboloid of revolution (**a**) and a Cassinian solid of revolution (**b**).

Suggestions for Playing the Flatland Game

The following suggestions will help teachers implement this game in the classroom:

- Teachers may introduce the game by providing the cross sections, perhaps using the clues in **figure 4,** and having the entire class work as builders.
- If students have drawn images of the real objects that they have physically sliced, they have already had practice in their roles as radiographers. Otherwise, have them practice before they play the game.
- We are not really seeing the Flatland view of the objects—this view is too hard to comprehend even for Flatlanders. The slices are drawn as a top view (as they would appear from above).
- The radiographers should be constrained somewhat in their choice of a solid; otherwise, their objects may be too complicated. For example, a swarm of mosquitoes passing through Flatland is not a good place to start.
- The importance of the illustrating component should not be underestimated. Students' ability to render the slices often speaks directly to their abilities in geometric visualization. Encourage students to make faithful slices but also recognize their varying artistic abilities.
- As the radiographers work in a group, they should be encouraged to talk among themselves and, discreetly, to use actual three-dimensional objects to help them visualize the slices.
- Because the exact nature of these developmental activities is different for the two different roles, encourage students to exchange roles in subsequent games (the builders would become radiographers).

SLICEFORMS: PHYSICAL THREE-DIMENSIONAL RECONSTRUCTION

With success in the roles of radiographers and builders in the Flatland game, students develop the ability to navigate the dimensional cycle shown in **figure 3**, an important developmental step. Nonetheless, students' success is largely qualitative. Neither the drawings nor the reconstructed solid is exact. In the game, students seek only approximately correct cross sections. Having done this, they are prepared to navigate the dimensional cycle more exactly with *sliceforms*—real, scaled, three-dimensional models of everyday objects.

Several sliceform models are shown in **figures 5** and **6** (the latter are original models designed and built by the authors' students). Sliceforms are constructed from sequences of perpendicular cross sections that have been notched with slots and then joined together in a regular array. Operating much like the inserts that protect beer bottles in a case from breaking, these card stock models are hinged so that they can collapse to be completely flat. These magical collapsing objects were invented by the Danish mathematician Olaus Henrici in the late nineteenth century and have gained attention through the popularizing work of John Sharp (1995, 2004).

Sliceforms enable students to take the Flatland game one step further. Now they reconstruct three-dimensional solids from their cross sections. Our goal is to design and then construct sliceform models of real-life solids.

As a first step, students should create a few sliceforms from stock models to help them understand the design and construction concepts. We include templates for one stock model, a lightbulb (see **figs. 7a** and **7b**). Many others can be found in *Sliceforms: Mathematical Models from Paper Sections* (Sharp 1995).

Sliceforms are constructed from medium-weight card stock on which the templates have been copied or printed. Have students cut around the outline of each slice. The slices labeled with an x are cross sections of the solid along an x-axis. Analogously, the slices labeled with a y are cross sections of the solid along a perpendicular y-axis. After students cut out each cross section, the next step is to carefully cut slots along the indicated lines. These slots are not slits. Students must remove a thin section of card stock to make a slot; otherwise, the sliceform will not collapse properly. Sharp scissors are essential for this step. After students complete several slices, they can begin to assemble the model. Slices marked x and y meet at right angles with their respective slots to form a square lattice when viewed from above. The slices are numbered sequentially so that their order is clear.

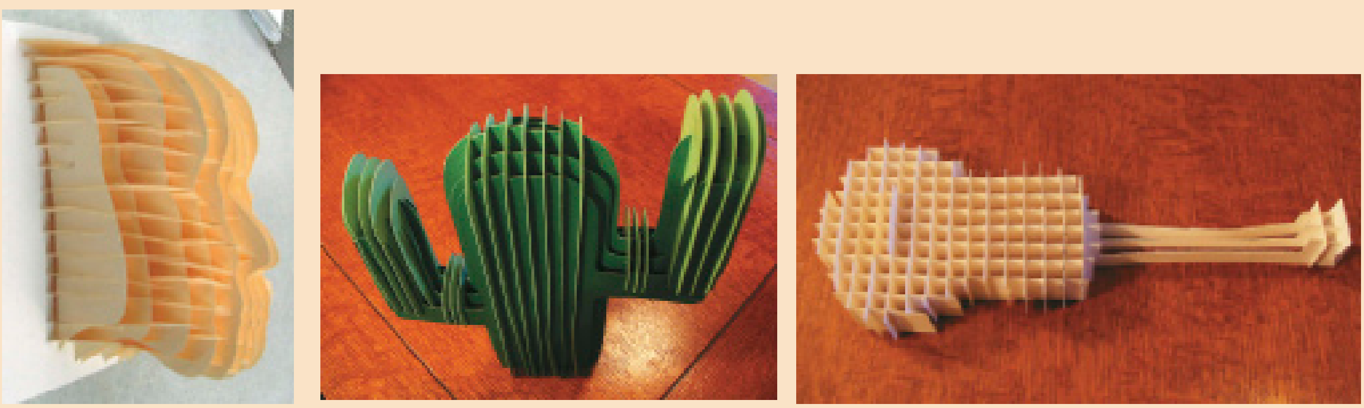
It should take students about twenty minutes to construct the stock model we have provided. When students are finished, they will see what a mechanical marvel these models are. They should also begin to have a sense of their many connections to geometry.

In contrast with assembling a sliceform from a template, designing and creating one's own sliceform of an everyday, solid object is a more significant undertaking. This activity will likely take the equivalent of several periods of class time, although much can be done at home. Students are captivated by this activity, and their increase in understanding of the dimensional cycle can be seen as they work.

Creating Sliceforms

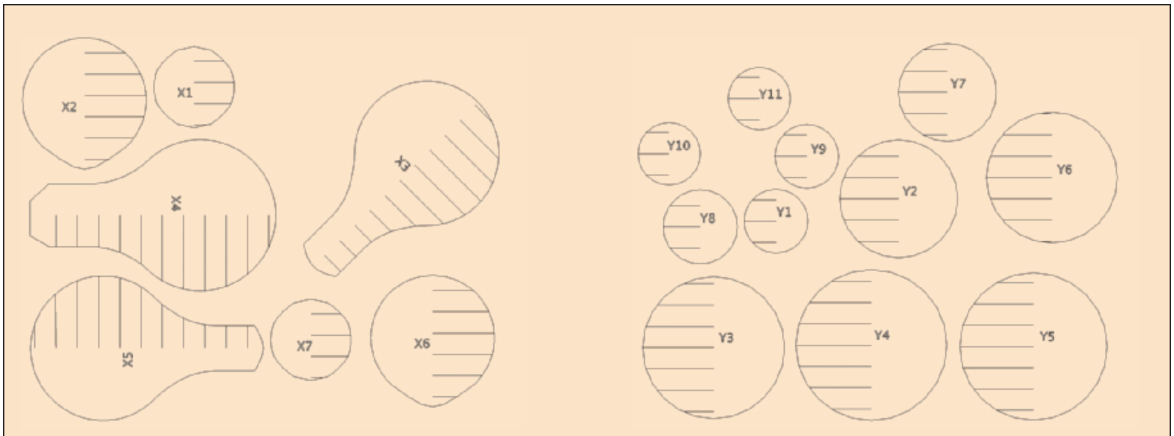
Following are a number of guidelines for students as they create sliceforms:

- *Importance of design.* Students should be encouraged to spend a significant amount of time at the outset on design. Poor design often cannot be overcome in the construction phase.
- *Experimental design.* Have students experiment. They will need to take many laps around the dimensional cycle as they design. Encourage students to use Play-Doh to shape their solid and then to slice mock sliceform cross sections. Bring in profile or contour gauges for them to use. In addition, have students use the free, three-dimensional design tool Google SketchUp to build mock versions. (The authors' website, www.westfield.ma.edu/ecke/flatland, includes instructions and a .skp file that will help students convert Google SketchUp models into sliceform templates.)
- *Slice regularity.* Slices should be made at regular intervals and come together at right angles. Graph paper of an appropriate size can be useful in helping students visualize an aerial view of their sliceform as well as the layout of their slices.
- *Problem solving.* The appropriate shapes and sizes of individual slices often lead to geometric problems that are perfect for consideration in mathematics classes. For example, the widths of successive vertical cross sections of a cylinder can be computed using the Pythagorean theorem.



(a) (b) (c)

Fig. 6 Some of the best original student sliceforms included "The Face" (**a**), "Cactus" (**b**), and "Guitar" (**c**).



(a) (b)

Fig. 7 These horizontal (**a**) and vertical (**b**) cross sections of a lightbulb can be used for the sliceforms activity.

- *Guess and check.* In the design of some objects—for example, a coffee cup—the slices can be determined fairly rigorously in both directions. In the design of other objects—for example, a rainbow trout—the slices in one direction can be made in a fairly obvious way, but it then becomes quite difficult to determine how to make the perpendicular slices correspond precisely. One strategy is to make the perpendicular slices too large on purpose and then cut off the excess once they have been integrated.
- *Depth of slots.* Students often have trouble determining the depth of their slots. Because the two integrated pieces have the same height along their intersection, a good rule of thumb is to have each of their slots take up half this height (see, e.g., **figs. 7a** and **7b**).

CONCLUSION

The activities introduced here offer dynamic, developmentally important ways to meet a number of crucial goals set forth in NCTM's Geometry Standards. In addition, the depth of the connection to contemporary applications of obvious importance does a great deal to motivate and inspire students. They see that specialists in areas such as medicine, engineering, art, digital technology, and architecture continue to make profound contributions based on the interplay between two-dimensional and three-dimensional worlds.

By developing abilities to navigate the world of three-dimensional objects in their full diversity, students may be able to contribute to new advances. Beyond this, perhaps the explorations here will inspire students to break free of the prejudices of our human dimension and help us learn to visualize exotic, higher-dimensional objects.

ACKNOWLEDGMENTS

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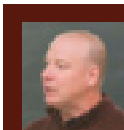
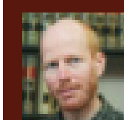
ANSWERS TO THE FLATLAND GAME

Answers are from clues given in **figure 4**: (**a**) cube or right rectangular prism; (**b**) lightbulb; (**c**) cone.

JULIAN F. FLERON, jfleron@westfield.ma.edu, and VOLKER ECKE, vecke@westfield.ma.edu, are colleagues at Westfield State University in Massachusetts. This article is a sample of their work on discovering the art of mathematics (see http://artofmathematics.westfield.ma.edu/), a project focused on developing a large collection of inquiry-based materials in mathematics.