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A Balancing Act: Making Sense of Algebra

Students think and act like mathematicians while using balance scales and bar diagrams and, most important, mathematical reasoning.

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We recently asked a group of middle school teachers, many of whom taught algebra, to answer the question, "What is algebra?" After a pregnant pause, we heard, "solving equations," "determining a value for the unknown," "using variables," and "graphing equations." Although these answers are all part of doing algebra, this branch of mathematics encompasses much more.

For most students, algebra seems like a totally different subject than the number topics they studied in elementary school when in reality the procedures followed in arithmetic are actually based on the properties and laws of algebra. In fact, algebra should be a logical next step for students in extending the proficiencies they developed with number because "algebra is simply a language for exploring and explaining mathematical patterns" (Bressoud 2012, p. 1).

In *Adding It Up: Helping Children Learn Mathematics*, the National Research Council delineates two aspects of algebra as being "(a) a systematic way of expressing generality and abstraction including algebra as generalized arithmetic; and (b) a guided transformation of symbols such as we do when we solve equations by collecting like terms and using inverse operations" (NRC 2001, p. 256). These transformational aspects have traditionally been emphasized in such a way that algebra becomes a study of procedures and rules rather than an exploration of concepts that lead to generalizations that support the rules or make the equation or expression meaningful to the student. Research has shown that these rule-based approaches to teaching and learning lead to forgetting the rules (e.g., Kirshner and Awtry 2004), unsystematic errors (e.g., Booth 1984), reliance on visual cues (Kirshner 1989), and poor strategic decisions (e.g., Wenger 1987).

Table 1 Students in various grades had different responses to make the sentence $8 + 4 = \square + 5$ true.

Responses	Grades		
	1 and 2	3 and 4	5 and 6
7	5%	9%	2%
12	58%	49%	10%
17	13%	25%	21%
12 and 17	8%	10%	2%

Source: Adapted from Falkner, Levi, and Carpenter 1999, p. 132

We need to help students develop and make sense of the rules they are using and show them how to employ a variety of strategies to solve algebraic problems. We also need to help students see algebra as generalizing computational procedures and operations they use with numbers. The Common Core State Standards for Mathematics (CCSSM) for middle school advocate for this in its discussion of expressions and equations for grade 6: "Apply and extend previous understandings of arithmetic to algebraic expressions" (CCSS 2010, p. 41). Activities for students should include opportunities to make sense of problems; reason abstractly; model their thinking using graphs, tables, diagrams, and so on; look for and make use of structure; and create arguments to justify their thinking and critique the reasoning of others. These are, in fact, four of the Common Core's Standards for Mathematical Practice (SMP).

We had an opportunity to work with students across the middle grades implementing a curriculum that encouraged students to think and act like mathematicians, thus using the Mathematical Practices consistently. We found that a deliberate emphasis on the Mathematical Practices while learning CCSSM content helped our students gain a much deeper understanding of concepts and procedures and the ability to generalize these using algebraic reasoning and notation.

EXPLORING EQUALITY AND BALANCE SCALES

We began in grade 6 with the concepts of equality and balance that are central to the study of equations. Research has shown that students need help constructing meaning for equality (e.g., Falkner, Levi, and Carpenter 1999; Kieran 1981; Saenz-Ludlow and Walgamuth 1998). **Table 1** shows how students in grades 1 through 6 responded when asked what number they would put in the box to make the sentence $8 + 4 = \square + 5$ true.

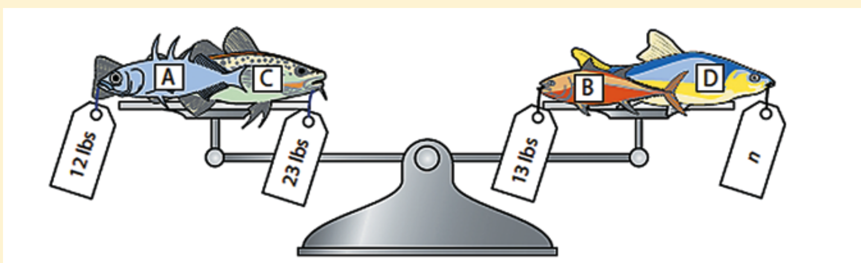
Note that a lower percentage of students in grades 5 and 6 (only 2 percent) got the correct answer compared with those in grades 1 and 2 (5 percent).

In the following investigation, students harkened back to ancient Egypt and imagined working in a fish market. They were shown the balance scale in **figure 1** and asked to write an equation for the weights of the fish on the scale. We asked them to use *n* in their equation to represent the unknown weight of the fish in pounds.

We also asked them to solve the equation for the missing weight without finding the total weight of the fish on the left. We introduced students to writing equations at this point but not to the typical rules in solving them. After giving students a chance to write their equations and think about how to find the weight of fish D, teachers and students engaged in the following conversation:

Ms. Jackson: KayAn, what did you write for your equation?
KayAn: I wrote $12 + 23 = 13 + n$. I could see that the fish on the left balanced the fish on the right. I used equals to show they were the same.
Ms. Jackson: Jason, can you repeat what KayAn said?
Jason: I think she said that she wrote $12 + 23 = 13 + n$ 'cause that is what I wrote. You can see the balance on the scale.
[Other students nodding.]
Ms. Jackson: It looks like we agree on

Fig. 1 Students were asked to write an equation and find the weight of the unknown fish in this diagram.



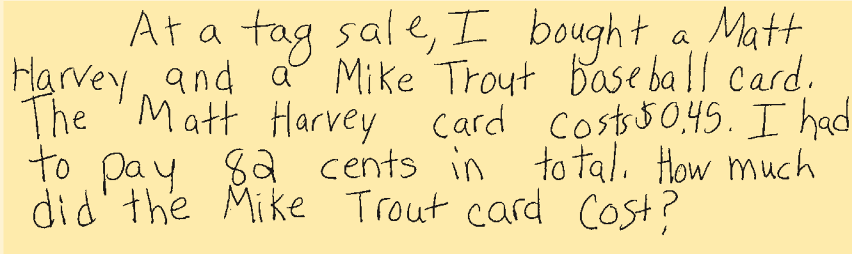
Source: Sheffield, Chapin, and Gavin 2010, p. 14

Fig. 2 This bar diagram illustrates another model to represent the fish problem.

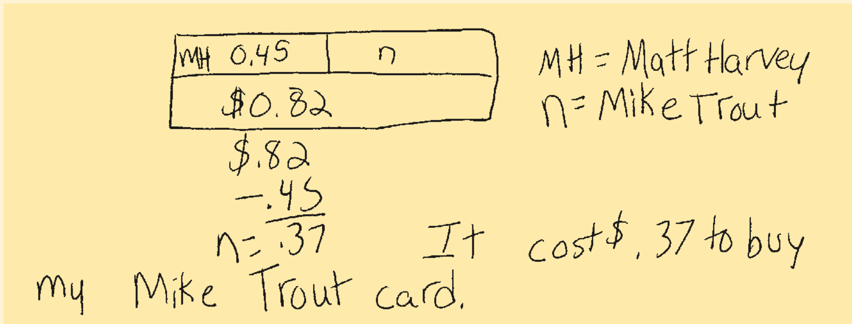
12 pounds	23 pounds
13 pounds	*n* pounds

Source: Sheffield, Chapin, and Gavin 2010, p. 15

Fig. 3 These examples of students' independent work showed a solidifying of their understanding of solving linear equations.



(a) Skyler's word problem for the equation $0.45 + *n* = $0.82



(b) Skyler's bar diagram, used in the solution to his word problem

(c) Kerry's solution to Skyler's problem using a balance scale

the equation. How did you solve it, Sarella?

Sarella: I saw 13 is 1 more than 12, so I knew that *n* would be 1 less than 23.

Carey: If 13 is 1 more than 12, why isn't *n* 1 more than 23?

Sarella: Well, see . . . you have to make up for it by taking 1 off the 23.

Ms. Jackson: Everyone turn to your elbow partner and talk about Sarella's solution and Carey's question. Should you add 1 to 23 or subtract 1 from 23?

After a one-minute discussion, Ms. Jackson asked Jared to share his answer. He drew blocks on a balance scale on the white board. He took 1 block from the pile of 23 blocks on the left side of the scale and moved it to the pile of 12 blocks to make 13. He then had one pile of 13 blocks and one pile of 22 blocks on the left side of the balance scale. He said that the pile of 13 blocks on the right side had to match the pile of 13 blocks on the left, leaving the unknown pile to be 22 blocks. Agreement was reached that this made sense.

Ms. Jackson: Did anyone solve it another way?

DeShawn: I saw that 23 blocks is 10 more than 13. That meant *n* is 22.

Susana: Where did 22 come from?

DeShawn: [Pointing to the fish] See this fish B with 13 pounds; it is 10 pounds less than this fish C. That means that fish D has to make up by adding 10 pounds. See, it has to be 10 pounds more than A or it won't balance.

Students figured out the solution without using pencil or paper. They made sense of the problem, explained and justified their methods, critiqued one another's reasoning, and gained a stronger grasp of the meaning of equality and how to interpret the equals sign (SMP 3).

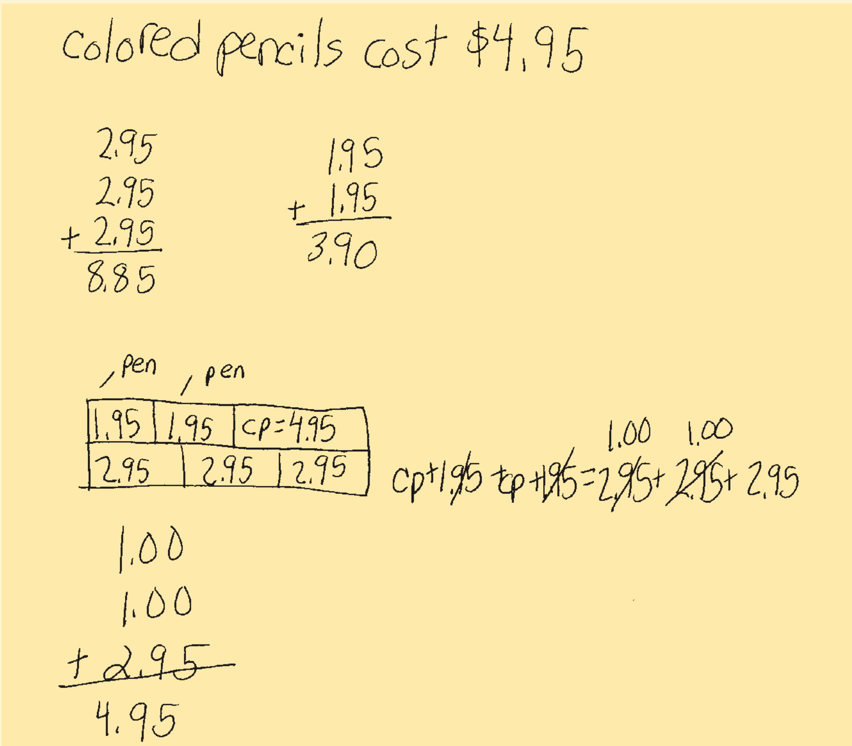
PROPOSING BAR DIAGRAMS AS EFFECTIVE TOOLS

After solving a few more fish-weighing problems, students were exposed to a different solution method. In the example that we posed, we stated that Mei Ling had drawn a bar diagram, shown in **figure 2**, to solve the problem. We asked students to explain how Mei Ling might have used this diagram to find the unknown weight.

Some students found this visual easier to understand than the balance scales. They saw that the total of 13 pounds was a little more than 12 pounds, so they needed to subtract from 23 pounds to get the value of *n*. English language learners particularly benefited from this visual model. Throughout the investigation, students were making sense of the problem (SMP 1) and using a different model to represent the problem (SMP 4). In fact, using these

Fig. 4 Dave explored finding the cost of a set of colored pencils.

Notebooks cost \$2.95 each, and 2 pens cost \$1.95. A set of colored pencils and 4 pens have the same cost as 3 notebooks. How much does a set of colored pencils cost?



Mathematical Practices helped them think and act as mathematicians.

USING A VARIETY OF STRATEGIES AND MODELS

We found that students needed to be able to transfer their learning to other applications and should have opportunities to do so during the learning process. Therefore, students were given a variety of contextual problems to apply these strategies to other situations. After solving and discussing several problems, students worked independently using balance scales, bar diagrams, and mental computation. They also produced contextual problems that would fit a given equation. Skyler created the situation in **figure 3a** for the equation $\$0.45 + n = \$0.82$. He then drew a bar diagram and solved the equation (see **fig. 3b**).

Kerry used a balance scale and reasoned that since \$0.45 and \$0.40 would equal \$0.85, she needed to subtract 3 cents from 40 cents to get her answer of \$0.37 (see **fig. 3c**).

We then posed more challenging problems to students, including the problem in **figure 4**. From Dave's work in **figure 4**, we see that he used a bar diagram and an equation with the given information to help solve the problem. He reasoned that since each notebook is \$1.00 more than each set of 2 pens, then the colored pencils must be \$2.00 more than the cost of the third notebook, which was \$2.95. Therefore, the colored pencils cost \$4.95. Giving students an opportunity to solve problems in different contexts helped them develop a deeper understanding of equations, the equals symbol (=), and the concept of equality.

Students next moved from contextual to symbolic problems. When working with the scales and bar diagrams, students learned how to solve certain types of equations using compensation strategies. "If I add 3 to a number, I must subtract 3 from another number to maintain balance and equality." They were given the following problems and asked to apply what they learned to find the value of n just by reasoning about balance and equality.

$$4832 + 197 = n + 200$$
$$49 + n = 73 + 50$$
$$23 + n = 14 + 24$$
$$51 - n = 50 - 25$$
$$78 + 32 = 80 + n$$

This activity helped students build fluency with mental computation, an important skill that is a hallmark of mathematicians. They were also reasoning abstractly and quantitatively (SMP 2). For students who needed more challenge, we asked them to write their own equations, which could be solved using similar compensation strategies, and trade papers with partners to solve.

The equations that students encountered in these initial activities were designed so that they could solve them mentally. Bar diagrams and scales emphasized balancing both sides of an equation to find the solution. As students progressed through the unit, they used tables and flow charts to solve equations. They also learned how to solve using the traditional approach of inverse operations. In so doing, they had a variety of strategies from which to choose to find the solution.

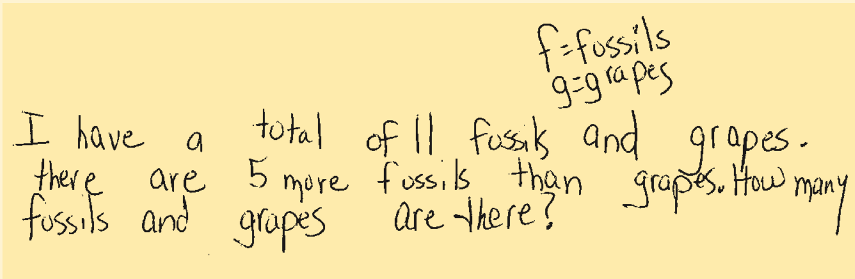
WRITING AND SOLVING EQUATIONS WITH TWO VARIABLES

Students then moved to the more challenging task of writing equations

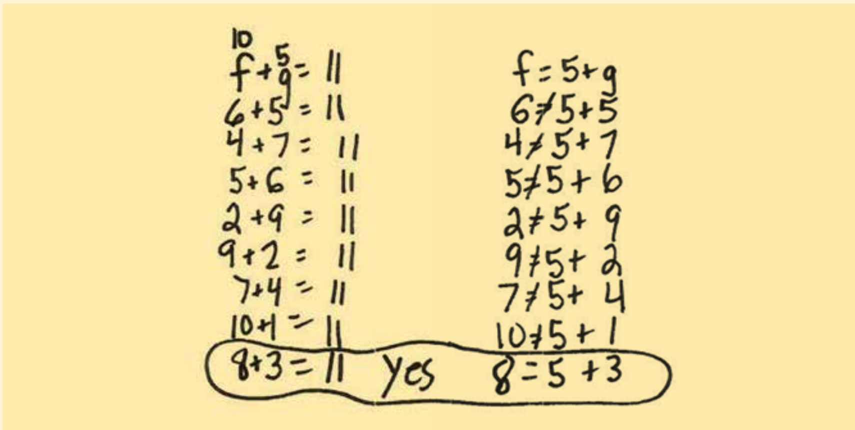
Fig. 5 Students produced a variety of riddles for systems of equations and solved them in different ways.

1a) Write a riddle that could be represented by the following equations. Explain what the variables stand for.

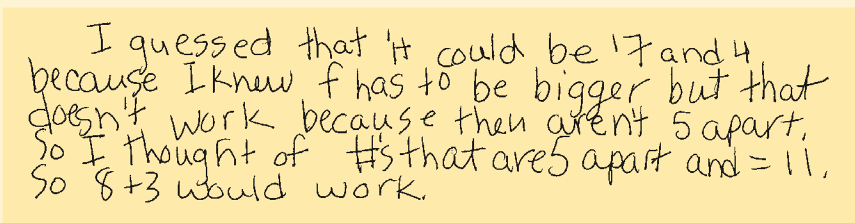
$$f + g = 11$$
$$f = 5 + g$$



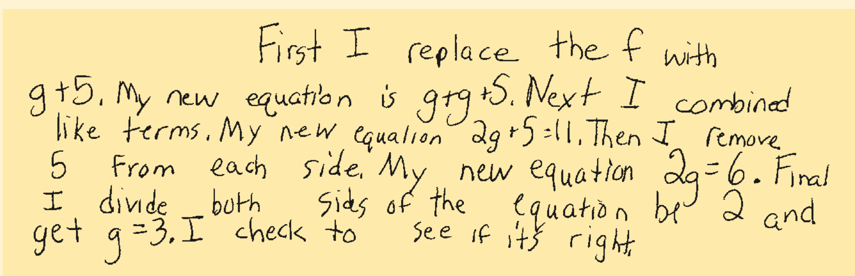
(a) A sample riddle



(b) A solution using guess and test



(c) Another solution using guess and test



(d) A solution that employed the replace, remove, and divide strategy

with two variables from contextual situations. This new situation required that they focus on the interrelationships among the variables as well as the effects of operations on the variables. In this investigation, students worked on writing such equations.

The activity began by setting the scene at the end of the Silk Road in ancient Egypt and writing an equation that stated the relationship between two animals. For example, when looking at a chart that showed that the trader, Iris, had 20 pigs and 12 horses for sale, students learned that Iris wrote $p - 8 = h$. Her husband, Seth, wrote $h + 8 = p$. Students were then asked to compare and critique different equations to match the same situation and to write other equations of their own to show the relationships among the numbers of the animals being traded.

We deliberately introduced common misconceptions. In one problem, students were given the equation $p + 11 = g$, and they said that this must mean that there were 11 more pigs than goats. In another, Seth saw that Iris had written $2c = h$ and said that she was wrong because they did not have 2 camels and 1 horse but rather 6 camels and 12 horses. Lively discussions ensued as students struggled to make sense of the notation. It is challenging for some students, in particular English language learners, to state these relationships. They often mix up the variables and/or the operations. Exposing students to misconceptions and asking them to critique one another's reasoning (SMP 3) helped them solidify their own understanding.

To conclude the unit, students learned how to write and solve sets of two equations (see **fig. 5**) with two unknowns, exploring the relationship between variables in equations. In keeping with our emphasis on sense making, students often began with

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a guess, test, and refine method (see **fig. 5c**). They then learned how to use substitution to find the solution, which we call the replace, remove, and divide strategy (see **fig. 5d**). Instead of guessing, they replaced a variable in one equation with an equivalent expression found in the second equation. They then removed numbers to isolate the variable. Finally, they divided by the coefficient of the variable to find the value of the variable.

Note that the first student listed all possible combinations of 11 first, using a table to model her thinking, and then found which one worked with the other equation (see **fig. 5b**). A second student explained in words his guess-and-test strategy (see **fig. 5c**), and the third student explained step by step how she used the replace, remove, and divide strategy (see **fig. 5d**). None of the students were randomly guessing but rather they were using methods that logically led to the correct answer. Again, this provided evidence of students thinking and acting like mathematicians.

How do we know that these types of investigations help students learn? We wanted to find out. We administered open-response pre- and post-unit tests to 305 students. On average, student scores went from 4.63 to 12.23, with 98 percent of students making gains. At the beginning and end of the school year, we also administered an open-response assessment based on CCSSM items used in the Smarter Balanced Assessment Consortium's Mathematics Showcase Materials for grades 6 and 7. Students outperformed a comparison group (with effect sizes at grade 6 = 1.3 and grade 7 = 1.6).

In conclusion, we found that CCSSM-based algebra investigations with a focus on the Standards for Mathematical Practice were successful in helping students develop a much deeper understanding of equality and variables and their relationships in equations. We believe that this will give students a strong foundation on which to build algebraic concepts as they progress through middle school and into high school.

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Any thoughts on this article? Send an email to **mtms@nctm.org.**—*Ed.*

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