7 DEVELOPING EFFICIENT COMPUTATION WITH MINILESSONS

I think, therefore I am. . . . Each problem that I solved became a rule which served afterwards to solve other problems.

—*René Descartes*

Mathematics is the only instructional material that can be presented in an entirely undogmatic way.

—*Max Dehn*

MINILESSONS WITH MENTAL MATH STRINGS

"I saw three groups of four, and there's two rows of them, so I know it's twenty-four." Deena, a New York City fifth grader, is explaining what she has just seen. Risa Lasher, her teacher, is beginning math workshop, as she normally does each day, with a short ten- or fifteen-minute minilesson focusing on computation strategies. Using an overhead projector, Risa has displayed, for a few seconds, the image shown in Figure 7.1. The children, who are all sitting on the rug in the meeting area with her, have been asked to talk to their neighbor about what they have seen.

Suzanne, sitting next to Deena says, "I agree, but I saw them as two groups of three on top and two on the bottom. That was my twelve. But then I doubled twelve, too—because it repeated." Darius and Sasha, on the other hand, have counted the groups of three and multiplied, 3×8; others have calculated the total differently still.

Risa starts a whole-group discussion. "Deena and Suzanne, I overheard you two saying that you saw two groups of four times three." On the overhead transparency Risa writes $2 \times (4 \times 3) = 24$. "Does this represent what you saw?"

Deena and Suzanne agree. "But we each saw the four times three differently," Deena explains as she goes to the chalkboard where the image is again being projected. She draws a circle around the top row, which she has used;

and then with a different-color chalk, she circles the four groups Suzanne has used.

"And Darius and Sasha, how did you two see it?" Risa continues the conversation.

"We just saw eight groups of three," Sasha responds for both of them.

Risa writes $8 \times 3 = 24$ and then asks, "Any other ways? Juan?"

"I saw two threes as a group of six," Juan explains, continuing with how he then just did 4×6. Risa writes the equation $4 \times (2 \times 3) = 24$ on the overhead transparency as well and ensures that what she has written represents Juan's thinking.

He agrees that it does, so Risa segues into a discussion of fractions. "So there are a lot of ways to calculate the whole here. What if I just circle these three? Could we think about this amount as a fraction? What part of the whole is it?"

"If the whole is twenty-four, then what you circled is three twenty-fourths, three out of twenty-four," offers David, proudly showing what he knows about fractions.

"But there are other ways, too," Deena asserts. "It's also one eighth."

"Why?"

"Because if you count each group of three as one group, then it is one group out of eight," Deena explains clearly.

David at first looks surprised but then is intrigued. He is now seeing both perspectives. "What is it one eighth of, David?" Risa asks, attempting to deepen his understanding.

"It's one eighth of—twenty-four?" David responds a bit tentatively.

"What do the rest of you think? Do you agree with David?"

Marta, who has been rather quiet until now, raises her hand and offers, "I agree with him. It is one eighth of twenty-four—because that is three—because twenty-four divided by eight equals three."

"You've said a lot there, Marta," Risa responds. "Let me write down all the equations that have been mentioned so far and let's see how they are related." Risa turns off the overhead, erases the board, and writes the following string of equations:

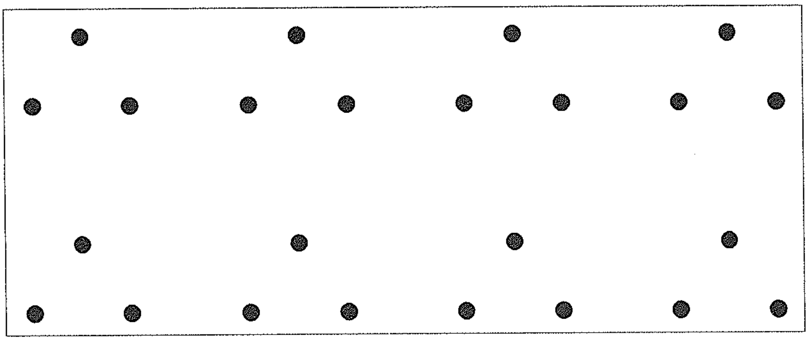


FIGURE 7.1 *Groups of Dots*

$$\frac{3}{24} = \frac{1}{8}$$

$$\frac{1}{8} \text{ of } 24 = 3$$

$$\frac{3}{24} \text{ of } 24 = 3$$

$$\frac{24}{8} = 3$$

$$8 \times 3 = 24$$

$$4 \times 6 = 24$$

$$2 \times 12 = 24$$

Then she suggests, "Turn to the person you are sitting next to and talk about the relationships you see in this string of equations." Risa uses this technique so that everyone will be involved; she hopes to get all her students to reflect on the *underlying big ideas.* She succeeds.

"What we talked about," Suzanne begins when Risa reopens the whole-group discussion, "is that they're related because multiplication and division are related."

"Tell us more. How so?" Risa probes.

"Well, like—there are eight groups of three. So that is twenty-four over eight equals three, but it is also one eighth of twenty-four equals three, because eight times three equals twenty-four."

Maria, who has been talking with Marta, chimes in, "Yeah, and you could do one third of twenty-four, too. That would be eight!"

Risa adds this equation to the string and asks, "Any other ideas?"

Paul, David's discussion partner, elaborates. "It's like the one is the three and eight is the whole. But twenty-four is also the whole—so it's one eighth of twenty-four equals three, but it's also one eighth equals three twenty-fourths."

"Could we make more related equations for the last two problems, four times six equals twenty-four and two times twelve equals twenty-four?"

David offers $\frac{1}{4} \times 24 = 6$ and $\frac{1}{6} \times 24 = 4$. Juan suggests $\frac{4}{24} = \frac{1}{6}$. Risa adds these equations to the string and comments, "And there are more, too, aren't there? It's really cool how division and multiplication relationships help us think about fractions."

In this brief minilesson, Risa has succeeded in bringing to the fore some important big ideas: the connection between multiplication and division and the fact that although the whole may change, the relationship between the parts and whole must stay constant for fractions to be equivalent. Usually each day, at the start of math workshop, Risa chooses a string of four or five related problems and asks her students to solve them. Together they discuss and compare strategy efficiency and explore relationships between problems, just as Dawn did with her children in the previous chapter. Today, however, Risa helped her students generate a string of related problems rather than presenting one herself. Whether students generate the string or solve a given string prepared by a teacher, the relationships between the problems are the critical element.

Good minilessons always focus on problems that are likely to develop certain strategies or big ideas that are landmarks on the landscape of learning. We call these groups of problems *strings* because they are a structured series (a string) of computations that are related in such a way as to develop and highlight number relationships and operations. Designing such strings and other minilessons to develop computation strategies requires a deep understanding of number and operation; the choice of numbers and the models and contexts used are not random.

CHOOSING THE STRATEGIES, CHOOSING THE NUMBERS, CHOOSING THE MODELS

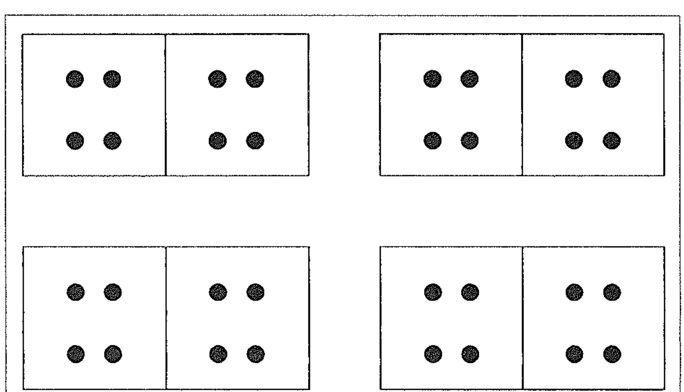
Finding a Familiar, Landmark Whole

The quick image that Risa chose was designed to prompt a discussion about the relationship between multiplication and division—an important idea by itself, but also one that will resurface later as children grapple with common denominators for addition and subtraction. It was pretty well guaranteed that Risa's students would see the dots grouped in a series of different arrays. Many other contexts (see Figures 7.2 and 7.3, for example) would likely generate similar discussions. Using the array in Figure 7.2, a teacher might ask questions similar to those Risa did, but here the number and groupings of the dots point up relationships between $\frac{1}{4}$, $\frac{1}{8}$, $\frac{8}{32}$, $\frac{1}{4}$ of 32, $\frac{1}{8}$ of 32, etc. With the money context in Figure 7.3, students are likely to talk about fractions and their decimal equivalents.

Let's return to Joel Spengler's sixth-grade classroom and observe a minilesson based on a string of previously prepared problems for adding fractions. Joel is using a clock model to encourage students to use the strategy of *finding a familiar, landmark whole*—in this case one hour.

"I live on the Upper West Side of Manhattan, near Central Park," Joel begins, "and I've been thinking that I should make use of the park and start exercising. I was talking about this earlier with my friend Alex, and he said

FIGURE 7.2



that I should start by running for one third of an hour and then walk for one fourth of an hour. How much time is that altogether?"

"Thirty-five minutes," Max answers quickly.

"How did you figure that?" Joel asks.

"Well, one third of an hour is twenty minutes, and one fourth of an hour is fifteen minutes. So that's thirty-five minutes."

Joel draws a clock on the chalkboard, marking the twenty minutes and the fifteen. "You just knew that one third of an hour was twenty minutes. What if one didn't know that? Could we prove it? Veronica?"

"There's sixty minutes in an hour. One third of sixty is twenty, and one fourth of sixty is fifteen."

"Okay, does everybody agree with that?" When everyone nods in agreement, Joel asks, "And so what portion of an hour is the thirty-five minutes?"

"Thirty-five sixtieths," Veronica replies quickly.

"It could also be seven twelfths," Lucy joins in. "You could think of it in five-minute chunks."

Joel adds the five-minute chunks to the drawing (see Figure 7.4) and writes $\frac{1}{3} + \frac{1}{4} = \frac{20}{60} + \frac{15}{60} = \frac{35}{60} = \frac{7}{12}$. Then he continues with his string. "Another friend of mine, Jesse, said I should push myself more and run for half an hour and then walk for two thirds of an hour." Joel writes the problem $\frac{1}{2} + \frac{2}{3}$ as he talks. "How much is that? Victor?"

"One hour and ten minutes—seventy minutes." Victor has the answer quickly, as do most of the other children.

"How did you think about it?"

"I thought of the half as thirty minutes," Victor explains, and Joel writes $\frac{1}{2} = \frac{30}{60}$. "Then I thought of the two thirds as forty minutes." Joel writes $\frac{2}{3} = \frac{40}{60}$. "So thirty plus thirty made an hour and I had ten minutes left."

FIGURE 7.3

"That's interesting," Joel comments, writing $^{30}/_{60} + ^{10}/_{60}$ under the $^{40}/_{60}$, "and what portion of an hour was that?"

Veronica responds, "The thirty sixtieths is equal to one half, so two thirds is equal to ten sixtieths plus one half, or forty sixtieths."

Chloe attempts to offer another strategy, but becomes confused. "I knew that ten sixtieths was ten minutes, too, so then I added three sixtieths to that."

"So did you get thirteen sixtieths?" asks Joel.

"No—no—I mean—one sixtieth?"

Joel focuses on the context to help her realize what she is trying to do. "Are you saying that ten minutes is one sixtieth of an hour?"

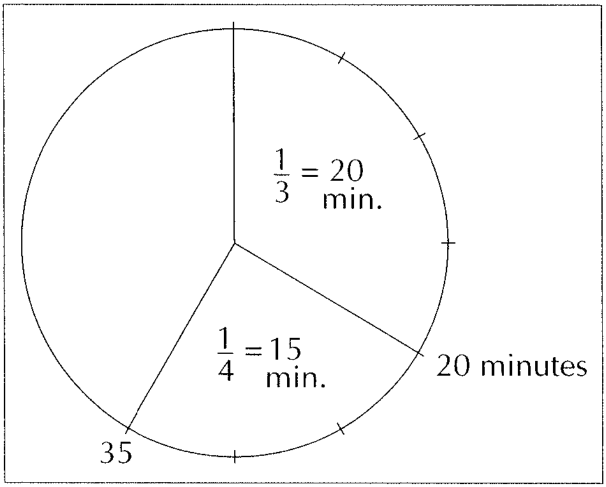
"Oh, no—I mean ten minutes is one sixth of an hour, and the half hour is three sixths, so that's four sixths." She clarifies her thinking for herself as she explains, and Joel marks the ten-minute chunks on the model of the clock as she talks it through. She seems clear, as do the other children, so he goes on with his string.

"Okay, so what if I run for one third of an hour, walk for twenty-five minutes, and then get a second wind and run for another quarter of an hour?" As he talks he writes $^1/_3 + ^{25}/_{60} + ^1/_4$.

"It's one hour; it's twenty plus twenty-five plus fifteen. Sixty minutes." Everyone agrees with Veronica's explanation, and Joel writes $^{20}/_{60} + ^{25}/_{60} + ^{15}/_{60} = ^{60}/_{60} = 1$.

Although Joel has thought about the problems beforehand and has a string of related problems ready, he does not put all the problems on the board at once. Instead he writes one at a time, and children discuss their strategies before the subsequent problem is presented. This way, the children can consider the strategies from the prior problem as well as the numbers, and they are prompted to think about the relationships of the problems in the string as they go along. Sometimes, depending on the strategies

FIGURE 7.4
Clock Model



he hears, Joel adjusts the problems in his planned string on the spot to ensure that the strategies he is attempting to develop are discussed and tried out. Let's watch him in action.

"Let's go back to one half plus one sixth for a minute. Chloe, earlier you said we could think of this as forty sixtieths, or four sixths, right? And, Lucy, you suggested we could also think of the clock in five-minute chunks. Could we use that here? Turn to the person you're sitting next to and discuss this." Joel wants the students to think about a variety of common denominators.

After giving the students a brief time to discuss how twelfths could be used and ensuring that everyone seems clear, Joel has Lucy share, and he writes $\frac{2}{12} + \frac{6}{12} = \frac{8}{12}$. Once again he uses the clock model, indicating the five-minute chunks. He now has this string of equations on the board: $\frac{40}{60} = \frac{2}{3} = \frac{4}{6} = \frac{3}{6} + \frac{1}{6} = \frac{1}{2} + \frac{1}{6} = \frac{2}{12} + \frac{6}{12} = \frac{8}{12}$. But he wants his students to consider further factors and multiples, so he asks, "We've been using a clock model to help us add fractions today. What made that a helpful model with these problems? Would this model be helpful with all numbers? For example, adding sevenths? Turn to the person you're next to and talk about this."

After allowing a few minutes for discussion and reflection, Joel starts another whole-group discussion. As a class they generate a list of fractions for which the clock model is helpful: twelfths, thirds, sixths, sixtieths, and fourths. They discuss why and then Joel concludes his minilesson with a suggestion that in the future when they are adding or subtracting these fractions, they might want to picture a clock and use it as a mental tool.

Choosing a Common Whole

Another day Joel uses the double number line to develop addition strategies.

"Let's start math workshop today with some work on addition," Joel begins. He writes $\frac{1}{4} + \frac{1}{5}$ on the chalkboard. "Is there a number we could use for say . . . an imaginary bike trip, that might help us calculate this problem? Hamilton?"

"One hundred," Hamilton is quick to respond.

"Okay. So I'll draw a line to represent the track and I'll write the one hundred miles at the end. How does this help? What did you do next?"

"I thought of it like percentages . . . out of one hundred," Hamilton explains. "Four times twenty-five is one hundred, and five times 20 is one hundred . . . so you just add up the twenty-five and the twenty . . . that's forty-five out of one hundred."

On the board, Joel represents his thinking on a double number line (see Figure 7.5) commenting as he draws, "So let me represent your strategy on the track. You knew one fourth of the trip was 25 miles, and one fifth of the trip was twenty miles. So how much of the trip have I done?

"Forty-five miles."

"Out of the one hundred, right?" Joel asks for clarification and then adds 45 to the bottom of the line, and the fraction, $\frac{45}{100}$, to the top of the line. Does everyone agree with Hamilton?"

Most acknowledge agreement, but Jeremy has his hand raised. "Jeremy, you have a comment?"

"Yes. I agree with Hamilton, but I did it a different way."

"How did you do it?

"I made the track twenty miles," Jeremy explains. Joel draws a new line and labels it *20* as Jeremy continues explaining, "I knew you had to do five times four because that is twenty. So one quarter of twenty is five."

Another classmate, Elise, interrupts. "I have a question first." She turns to Hamilton. "Why did you choose one hundred? Why not fifty . . . or whatever . . . ?"

"Because I was thinking about percentages."

"And so that made it easy for you?" Joel asks.

Hamilton nods, and Joel continues, "But now we'll try twenty. Alexis, you look like you want to finish the strategy that Jeremy was sharing." Joel tries to bring more voices into the conversation.

Alexis smiles, and explains, "One quarter of twenty is five and one fifth of twenty is four." Joel draws the double number line representation (see Figure 7.6) and Alexis continues, "So that's nine."

"Could I also write nine twentieths here?" Joel inquires pointing to where he has written $\frac{1}{4} + \frac{1}{5} = \frac{45}{100} = \frac{9}{20}$.

"Yes, because they are equivalent," Alexis offers. Several other classmates nod to show their agreement.

Joel pushes further. "Could we prove that?"

"It's like one track is five times as big," Hamilton offers. "Twenty times five is one hundred and nine times five is forty-five."

"Nice way of thinking about it," Joel comments and points out how he has multiplied the numerator by five, and the denominator by five. "Let's try

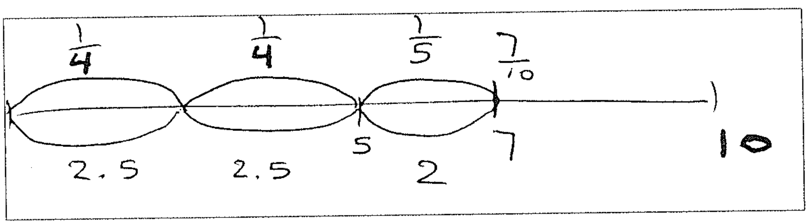


FIGURE 7.5 *Joel's Double Number Line Representing Hamilton's Strategy*

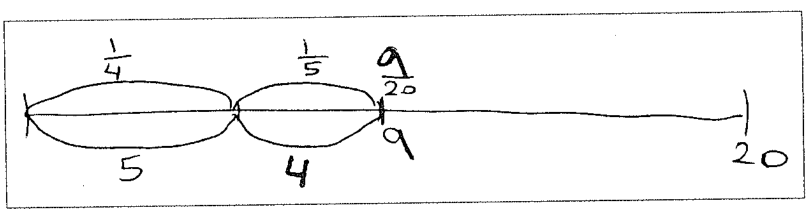


FIGURE 7.6 *Joel's Double Number Line Representing Jeremy's Strategy*

another problem and use this track idea again." Joel writes $\frac{2}{4} + \frac{1}{5}$ and continues to try to bring more children into the discussion. "Walker, do you have a good number for this one?"

"One hundred," Walker responds quickly. "Because I think of quarters. One quarter is twenty-five, but it's two quarters so that is fifty. One fifth is twenty."

"So what fraction of the course is it?" Joel asks for his final answer.

"Seventy hundredths, or seven tenths."

Joel writes $\frac{70}{100} = \frac{7}{10}$ and queries, "These are equivalent?"

Walker agrees, but Joel continues exploring the equivalence on the double number line, "Okay, then let's look at this as if the track is ten miles and see. How much is the one fourth?" he asks as he draws the line (see Figure 7.7).

At first, there is puzzlement, but then Hamilton responds, "Oh. Two and a half miles."

"So another fourth gets us to five out of the ten miles. And the fifth?" Joel records the result in miles underneath the line and the fractions above the line as he talks. "So the total miles is seven. And so the fraction is seven tenths?"

"Yes, but it almost seems like cheating. It's too easy. The seven and the ten are already there underneath the line," Hamilton says with a laugh.

Joel laughs back good-naturedly. "That's the point! We can make fraction problems friendly like this. It's like turning the fractions into friendly numbers so that we can do the computation mentally. We have two lines here that we can compare simultaneously, what's on top, and what's underneath. The top has the fractions. On the bottom we have the miles."

Whereas the clock model is limited to certain fractions, the double number line is an open model—any common denominator can be chosen and used. Initially the double number line is a model of the students' thinking, a representation Joel uses to enable everyone to visualize and discuss various strategies that come up. But as the students become more and more comfortable with it, it becomes a mental model—a tool to think with.

Multiplying Numerators and Denominators

The open array is a powerful support when multiplying and dividing fractions. In the following scenario, Maarten Dolk uses it with a group of inservice teachers.

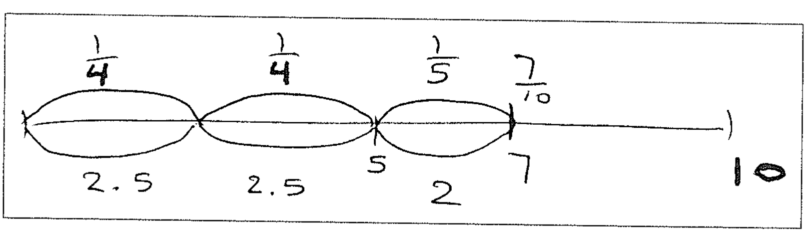


FIGURE 7.7 $\frac{2}{4} + \frac{1}{5}$

Maarten writes $\frac{1}{3} \times \frac{1}{4}$. Everyone quickly has an answer of $\frac{1}{12}$, and Maarten draws the open array in Figure 7.8a. Next he writes the problem $\frac{2}{3} \times \frac{1}{4}$.

"Well, that's just double the last one!" Marcia exclaims, noticing the relationships between the numbers Maarten is playing with.

"Tell us how you know that, Marcia."

"Because two thirds equals two times one third. So the answer has to be double."

Maarten draws the array in Figure 7.8b to represent her thinking and asks the group if they agree. When everyone does, Maarten writes the next problem: $\frac{2}{3} \times \frac{3}{4}$.

"Oh, so now it is just three times the last one!" Peter recognizes the relationship in Maarten's string.

"Why?" Maarten asks.

"Because three fourths is three times one fourth." Once again Maarten adds to the array (see Figure 7.8c).

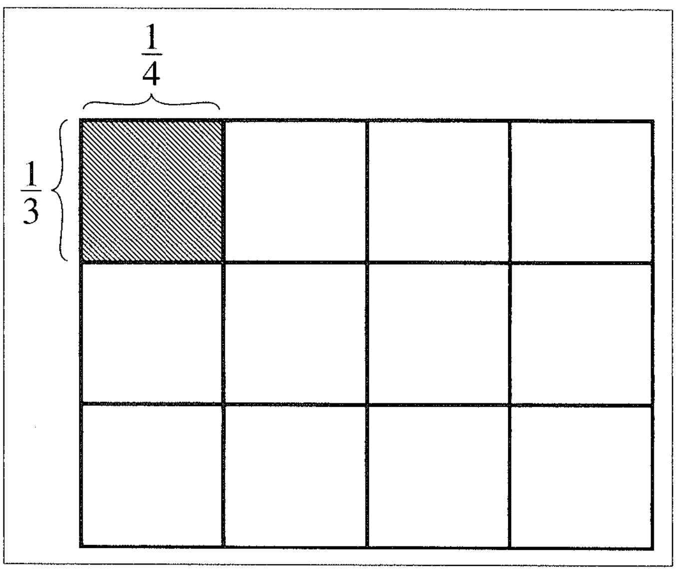
"Or another way you could think about it," Cleo offers, "is as two times three times one twelfth. You could use the first problem to build from instead of the second. So now it is six times one twelfth."

"Oh, my gosh, that's why the algorithm works!" Sandy exclaims. "I'm embarrassed to say, but I just memorized the procedure. I never knew why."

"So what are you noticing now? Can you use the array to tell us what you mean?" Maarten asks for more clarification; he wants to see whether she can relate the algorithm to the arrays he has drawn.

Sandy takes a moment to think, then comes to the chart and points to the first array. "This square is one twelfth. When you look at the last array,

FIGURE 7.8a
$\frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$



you see that the numerators form a two-by-three array—so now it's six times one twelfth."

"Oh, that's cool, Sandy," Marcia says. "It's also like the numerators form the inside array, and the denominators form the outside, larger one. So you can just multiply the numerators and multiply the denominators."

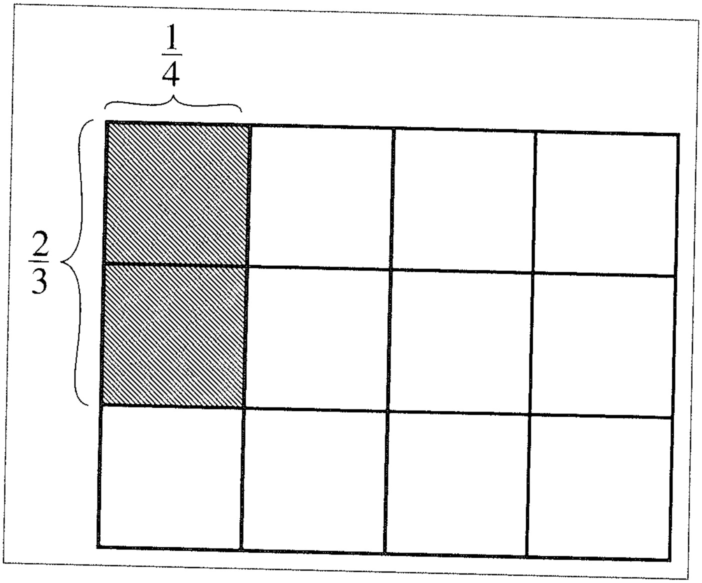


FIGURE 7.8b
$\frac{2}{3} \times \frac{1}{4} = \frac{2}{12}$

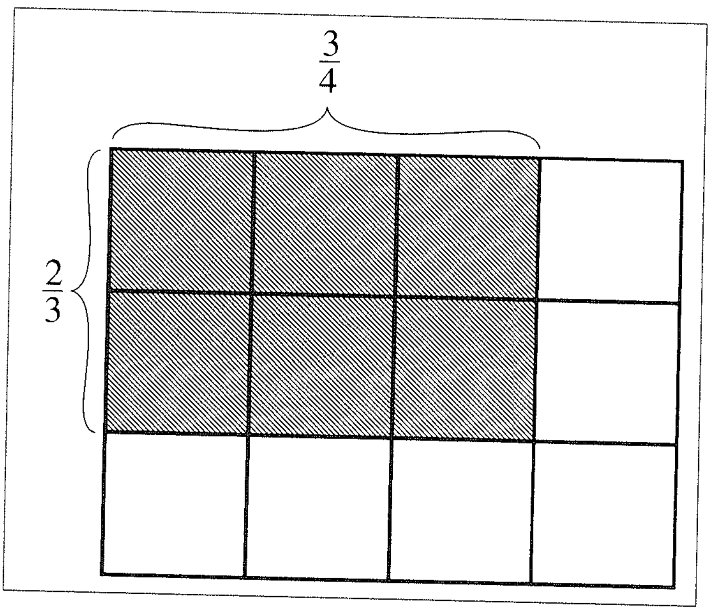


FIGURE 7.8c
$\frac{2}{3} \times \frac{3}{4} = \frac{6}{12}$

"What if the problem had been three halves times four thirds?" Maarten pushes.

Marcia pauses for a moment but then responds, "Oh, neat, now the numerators are the bigger array and the array made by the denominators is in the inside. And you can see how it fits inside twice!"

Maarten draws the open array to represent her thinking (see Figure 7.9), gives everyone a few moments to think about what Sandy and Marcia have said, and then comments, "Okay, so this time I'll put up two problems, instead of one. You make the open arrays to show how they are related, then turn to the person you are sitting next to and share your drawings." Maarten writes $1/5 \times 1/7$, then $3/5 \times 4/7$. As the class members work, Maarten moves around, looks at their drawings, and listens to their conversations.

Swapping Numerators and Denominators

Ensured that everyone seems to understand the ideas discussed thus far, he puts up the next problem in his string, $4/5 \times 3/7$.

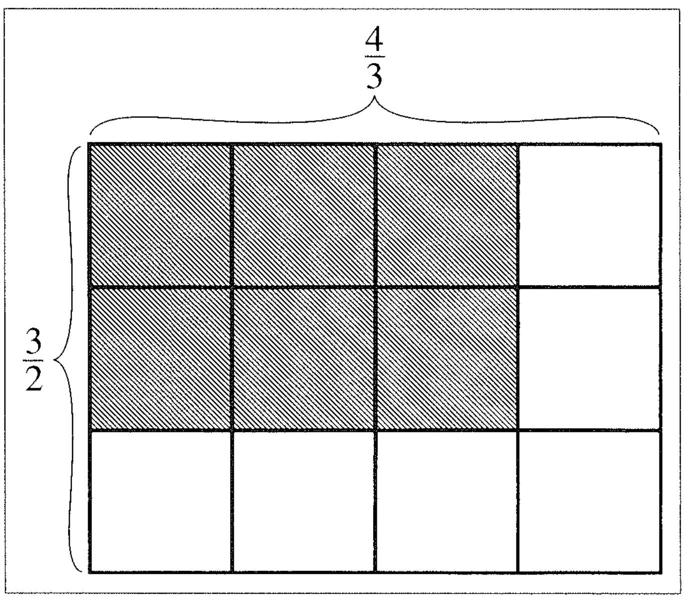
"Oh, wow! It's the same thing!" Carla is intrigued. "Does that mean you can always swap the numerators when you're multiplying fractions?"

"Or you could swap the denominators!" Peter exclaims.

"Why?" Maarten asks. "Turn to the person you are next to and talk about this."

Cleo and Roger are sitting next to each other. Maarten listens in on their conversation. Cleo is trying to convince Roger, who is still not sure it will always work. "It has to be so. The problem is really three times four times

FIGURE 7.9
$3/2 \times 4/3 = 12/6$
The Whole Is Six



one fifth times one seventh. Because of the associative property you can multiply whatever pieces you want first. You can associate them any way you want. It could be four times one fifth times three times one seventh or four times one seventh times three times one fifth, whatever you want."

Roger is still puzzled. I need to see it in the array." He begins to draw an array for $3/5 \times 4/7$ (see Figure 7.10a). "Okay, I understand this. It's a three by four inside a five by seven. So that is twelve thirty-fifths. Now I'll draw the other problem, four fifths times three sevenths." This time he draws the inside array as a four by three, but the outside array stays the same (see Figure 7.10b). "Oh, wow, look! The inside just turns ninety degrees, that's all. The relationship stays the same! So if we swapped the denominators—let's see—the outside array would just turn!"

When the whole-group discussion resumes, Maarten asks Roger to share what he and Cleo have been discussing. When he finishes, Sandy also has an insight she wants to share. "I see what you mean about the associative property being at play here. But I was also thinking that the commutative property is. One fifth times one seventh is one thirty-fifth. And that is also true if it is one seventh times one fifth. Since the numerators can also be swapped, it's either three times four or four times three. But either way it's twelve times one thirty-fifth—or twelve squares out of the thirty-five." She points to Roger's arrays. "Swapping *is* the commutative property. That's what it means—to *commute*."

"These would have been the next problems in my string, but you're all way ahead of me." Maarten smiles, pleased with their thinking, as he writes the following problems: $3/8 \times 4/6$; $5/6 \times 3/5$; and $4/5 \times 5/8$. "Let's see if we can figure out when this swapping strategy might be helpful."

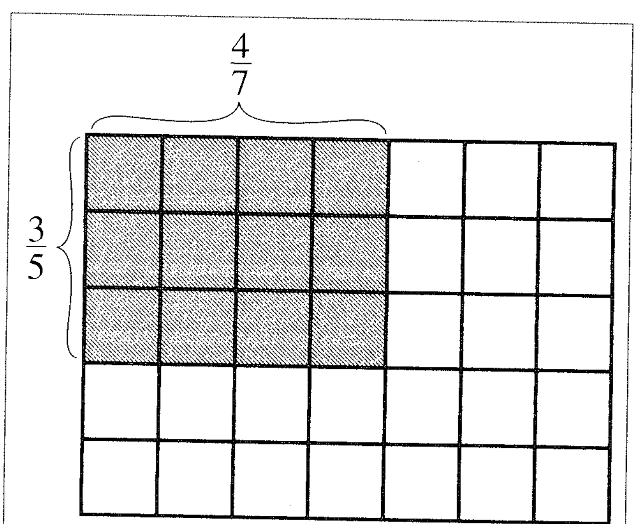


FIGURE 7.10a
$3/5 \times 4/7 = 12/35$

"Wow, cool." With oohs and ahs the group quickly transforms the problems into simple ones: $\frac{3}{8} \times \frac{4}{9} = \frac{4}{8} \times \frac{3}{9} = \frac{1}{2} \times \frac{1}{3}$; $\frac{5}{6} \times \frac{3}{5} = \frac{3}{6} \times 1$; $\frac{4}{5} \times \frac{5}{8} = 1 \times \frac{1}{2}$.

These teachers have of course constructed ideas relative to swapping and the multiplication algorithm much faster and more smoothly than middle school children will. The open array and strings Maarten used, however (the string in its entirety is shown in Figure 7.11), can be very helpful for children as well. Maarten chose the numbers in his string carefully: each problem is related to the one before it and to the strategies he wants to discuss.

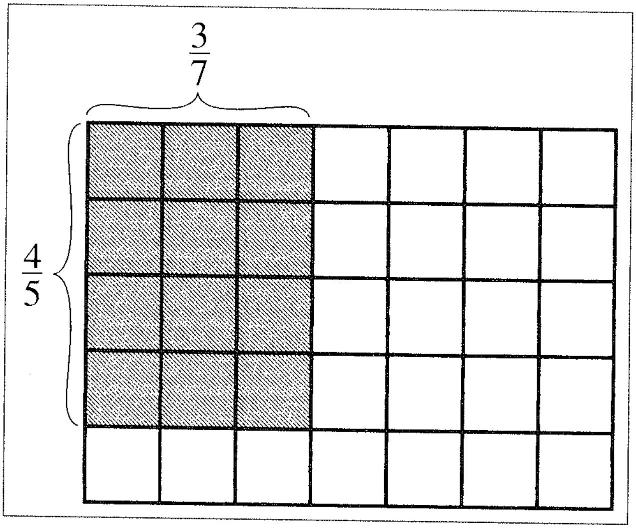
Getting Rid of the Fraction

The algorithm (multiplying numerators and denominators) and swapping are two important strategies for multiplying fractions. A third, *getting rid of the fraction*, is just as powerful. For example, if one wants to multiply $3\frac{1}{2} \times 18$, one can double the $3\frac{1}{2}$ to get rid of the fraction. If this amount is doubled, then the 18 needs to be halved. So now the problem is 7×9, one that is easily solved mentally. If the original problem was $3\frac{1}{4} \times 28$, we could get rid of the fraction by multiplying the $3\frac{1}{4}$ by 4. This gives 13. If we now take $\frac{1}{4}$ of 28 we get 7. So now the problem is 13×7. Once again we can solve it mentally. Try $3\frac{1}{5} \times 50$; the problem can be turned into 16×10!

To help children develop this strategy, just build a string of related problems (see Figure 7.12) in which several pairs of problems have the same answer. Children will quickly become intrigued with why the same answer occurs and a rich conversation and/or investigation is likely to ensue.

Getting rid of the fraction can also be helpful when dividing fractions. For example, if the divisor becomes one, the problem is solved! Take $3\frac{1}{3} \div \frac{1}{2}$. To get rid of the $\frac{1}{2}$ and make 1 we have to double it. With division we

FIGURE 7.10b
$\frac{3}{5} \times \frac{4}{7} = \frac{4}{5} \times \frac{3}{7} = \frac{12}{35}$



have to double the dividend, too, to keep the ratio constant. (Equivalent fractions, remember!) So now we have $6\frac{2}{3} \div 1$. Done! Or what if the problem had been $3\frac{1}{3} \div \frac{1}{3}$? Multiply both by 3, and now we have a very easy problem: $10 \div 1$! This idea, of course, is the basis of the invert-and-multiply algorithm. To make the divisor one, we can multiply by the reciprocal (in this case, $\frac{3}{1}$). Strings to explore this idea are shown in Figure 7.13. (Readers who feel lost here may want to refer to volume 2 of this series where we discuss

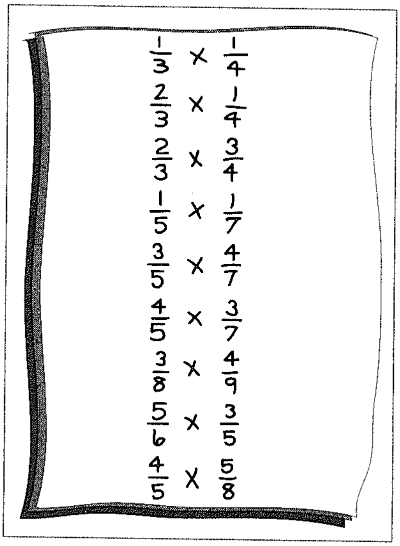


FIGURE 7.11
Maarten's String

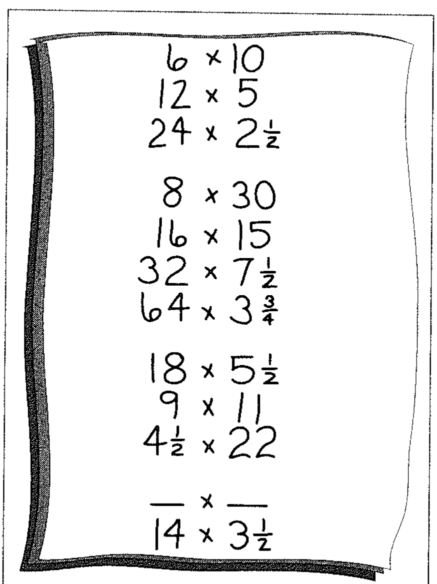


FIGURE 7.12
String in Which Several Pairs of Problems Have the Same Answer

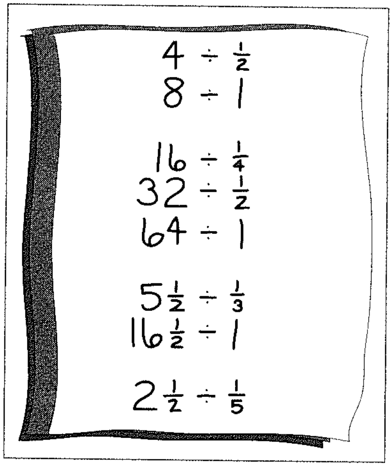


FIGURE 7.13
Strings to Explore Multiply by the Reciprocal

multiplication and division strategies in more depth, particularly the reducing strategy for division.)

Developing Strategies for Computation with Decimals

Volumes 1 (addition and subtraction) and 2 (multiplication and division) of this series discuss several strategies for the four operations with whole numbers. Because decimals are specific instances of fractions, and because they rely on our place value system, all these whole number strategies come into play. For example, compensation, a powerful mental math strategy for addition, can be used with decimals to make the problem friendly. Try adding 71.97 + 28.2. If we make the problem 71.17 + 29 by removing .8 from 71.97 and adding it to 28.2, we can get rid of one of the decimals—compensating in order to move to a landmark number. Or we could turn it into 72 + 28.17. Either way the problem has been made friendly enough to calculate mentally. Thinking about money also helps. One could think about adding 72 dollars and 28 dollars and then the difference of 17 cents. The open number line is very helpful here as a model to represent learners' thinking. (This is described in depth in volume 1.)

Using Money

Because money is such a powerful and ever present context in children's lives, it can be used to develop landmark numbers like .25, .50, and .75. Let's listen as Carol Mosesson's students discuss their multiplication strategies with decimals based on the use of money. Zenique is sharing how he came up with 1.80 as the answer to .20 × 9.

"Five twenty cents is one dollar," he begins, "so another four is eighty cents."

"Money is a clever way to think about that problem, Zenique." Carol writes .25 × 9 on the board. "How about this one? Shakira?"

"Three hundred—no, two-point-twenty-five. I counted by quarters."

"How many twenty-fives would it take to make three hundred?" Carol probes.

Shakira answers quickly, "Twelve."

Carol is surprised. "Wow, how did you know that so fast?"

"I knew four quarters made a dollar, so times three, that's twelve."

"That's exactly how I did it," Olana joins in. "I got two-point-twenty-five because I knew that four times twenty-five was a hundred, because that's like four quarters. So another four times twenty-five is another dollar. And one more quarter is two-point-twenty-five."

To develop this type of thinking, teachers can begin a minilesson using real coins, or pictures of coins, in an array. For example, if a four-by-four array of quarters is shown, many children will explain that they know that each row is a dollar. Because quarters are worth twenty-five cents, the problem can then be written as $16 \times .25$ and strategies developed—$16 \times .25 = 4 \times (4 \times .25)$, for example. After several minilessons based on money, children are able to use it as a context even when the numbers are "bare," as in strings.

Using Fractions and Decimals Interchangeably

Once children develop a sense of landmark fractions like ½, ¼, and ¾ and know the decimal equivalents, using them interchangeably is a powerful strategy. For example, 75×80 can easily be solved by thinking of the problem as $\frac{3}{4} \times 80$. You only have to remember to compensate for the decimal in the answer (multiply 60 by 100, because you treated 75 as $\frac{75}{100}$).

Minilessons can be designed to develop this ability. If the problems in a string progress from fractions to decimals to whole numbers, children quickly see the resulting patterns. For example, a string like $\frac{1}{4} \times 80$, $.25 \times 80$, 25×80, $\frac{1}{2} \times 60$, $.5 \times 60$, $.50 \times 60$, $.50 \times .60$ produces the appropriate patterns in the answers, and children can use arrays to explore the relationships.

SUMMING UP . . .

When René Descartes said, "Each problem that I solved became a rule which served afterwards to solve other problems," he said it all. When children are given the chance to compute in their own ways, to play with relationships and operations, they see themselves as mathematicians and their understanding deepens. Such playing with numbers forms the basis for algebra and will take children a long way in being able to compute not only efficiently but elegantly. Max Dehn envisioned the power of mathematical play when he said: "Mathematics is the only instructional material that can be presented in an entirely undogmatic way." Why has it taken us so long to realize it?