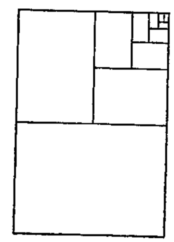
3B Organizing Shapes

"You can make a whole by adding one half to one fourth to one eighth to one sixteenth to one thirty-second to one sixty-fourth .... Well, you just keep doing that forever."

Kim's picture, showing how a process of assembling successive halves will lead to a whole unit. (More formally, this insight might be expressed as: $1/2 + 1/4 + 1/8 + 1/16 + 1/32 + 1/64 + \ldots = 1$)



This was part of Kim's explanation of a picture she had drawn in response to my (Brent's) prompt, What are some of the ways you can use your Fraction Kits to cover a whole page?

This event occurred about 10 days into our unit on fractions in Kim's Grade 3 mathematics class. I'd noticed Kim's interest in the process of successive halving right from the start of the unit — which began, as all my units in mathematics begin, with a few activities intended to provide learners with some shared experiences that could be interpreted mathematically. In this case, the starting place for our explorations of the patterns and relationships underlying fraction concepts was a series of paper-folding tasks.

I had first used such paper-folding activities several years earlier, and had found them to be effective tools to use when structuring units of study on fractions. In particular, paper-folding has served as a means to tap into my students' prior experiences with and knowledge of the topic.

When I began teaching mathematics, I often found it frustrating to structure tasks that drew on the diversities of understanding that were already represented in my classroom. I knew, for example, that my students' had all had rich and extensive experiences of cutting, subdividing, assembling, sharing, and so on. I was also aware that they arrived with at least a preliminary knowledge of fraction notation. As well, most came with good, although sometimes fragmented, understandings of various relationships among fractional amounts (e.g., 4 quarters makes a whole, a half is more than a third). The problem was that these experiences and insights were simply too diverse. I couldn't draw on them in ways that would be meaningful and useful to everyone.

A shared activity solved the problem. The common experience of folding gave us something we could all talk about, in the process highlighting what was already understood and what needed to be studied.

In this grade 3 classroom, we had spent the entire first session of our unit on an exploration of half-folds. With piles of scrap paper in the middle of each group's table, I asked students to fold pages in half in as many different ways as they could imagine. As might be expected, the first two folds that were made by every group were lengthwise and widthwise. These were followed by folds along diagonals and folds made by matching up opposite corners. But then there was a pause as groups wondered about other possibilities.

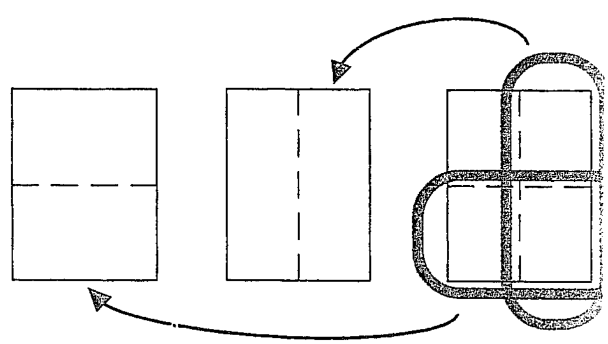
"Do the halves have to be the same size?" several students asked, opening the door to a brief exchange around the mathematical definition of one half and the less rigid interpretations of one half that often come up in day-to-day life. Even though the notion of bigger and smaller halves was familiar to everyone there seemed to be no difficulty appreciating that mathematics would not allow for such variations. Only equal parts were permitted when working with fractions.

That comment prompted a worry: Is a half that is made by folding a piece of paper lengthwise the same as a half that is made by folding it widthwise?

Everyone seemed to agree that they must be the same size. Wanting them to think more in terms of formal justification of such claims, I asked, "Pretend that you have a friend who thinks that the squarish half [i.e., folded widthwise] was larger than the longish half [i.e., folded lengthwise]. How would you prove to him or her that the two pieces were the same size?"

It took only a few minutes of discussion among themselves before Kim's group offered, "You could cut the halves in half. Both kinds of half can be made out of two half halves." That is, both a half-cut made lengthwise and a half-cut made widthwise can be shown to cover the same area as two identical fourths — an argument that most everyone agreed would convince an uninformed friend.

A **"proof"** that one sort of half is the same size as another sort of half: Combining the folds produces fourths. Each half piece can be made from two identical fourths.



With this demonstrated insight, I felt that the class was ready to examine more complicated folds. To start the next day's class I asked, "What would happen if I folded two times to make fourths ... and then folded again?

What I had expected was that everyone would answer, "You'll get eighths," thus setting the stage for my planned lesson on combinations of folds. What actually happened was that, although a handful of students were willing to argue that three half folds in a row would generate eighths, the majority of students who were willing to put forward their opinions felt that the product would be sixths.

I was surprised by this response, but I wasn't dismayed by it. It did, after all, show that these children were thinking in terms of patterns and relationships. One fold generated two pieces, two folds led to four pieces. It seems quite reasonable to expect that three folds would result in six pieces, four would generate eight, and so on.

A quick experiment to check the hypothesis demonstrated that a different pattern was at work, though. It wasn't that each successive half fold increased the total number of sections by two, but that each such fold doubled the number of pieces.

Kim was among the most vocal in expressing her surprise at this result, but seemed to resolve the issue to her satisfaction by the end of the day. Her journal entry for that class included a comment about the event, noting that "every part gets folded" — that is, that the number of sections doubles when a new half-fold is made. Alongside she drew a series of diagrams, showing a sequence of folded pages (which very much resembled the diagram of the "Fraction Kits," presented on page 137).

Perceiving some potential for the added development of the concept at hand, I pressed the issue a bit further. How many pieces would there be if we made another half-fold? If we did it a fifth time? What if we did it ten times?

I set the students to work on these tasks — and the second surprise of the lesson occurred. I expected everyone to continue the doubling pattern that I thought I'd just highlighted. But, to my seeing, only a few students took this tack. Everyone else began to fold and unfold, count and recount, assigning one another the tasks of determining the totals for four or five or ten folds.

It took only a few minutes for frustration to set in. The pieces of paper began to refuse the creases and the folded sections became too numerous to count accurately. In the hope of assisting students in their efforts, I drew a chart on the board. Although my initial intention was simply to provide a means to collect the emerging responses, I realized that this recording tool could also be used as a generative device as I was drawing it. That is, the chart proved useful in helping learners notice and extend the pattern that was at work here.

Number of half-folds	Number of sections
0	1
1	2
2	4
3	8
4	16
5	32
6	64
7	128
8	256
9	512
10	1024
⋮	⋮

The **filled-in chart** showing the relationship between the number of folds and the number of parts.

The pace at which the number of pieces increased was surprising to many. Even after a quick discussion of the logic behind the doubling pattern (and a double-check on the calculations), a few students announced their doubt. Given that the paper refused to cooperate beyond six or seven folds, a pair of skeptics (Kim was one of them) took out pencil and ruler and began to draw in the folds rather than actually making new creases. Though less-than-perfectly divided, a page covered with lines that marked out 1024 "sort of" equal parts was soon ready for display. In the meantime, others in the class experimented with extending the pattern with larger pieces of paper.

Happy with the thinking that I saw happening, I prepared a few questions that I thought might help to extend the investigations: What would happen if we did third-folds instead of half-folds? How many half-folds would it take until you were in the millionths range?

The first question didn't take. The second one, however, generated a great deal of interest, and the balance of that class was spent in folding, cutting, drawing, and shading efforts, all aimed at isolating "about one millionth" of a sheet of newspaper (which turns out to be slightly smaller than 1 mm x 1 mm) and about one millionth of one panel of the chalkboard.

The math lessons over the next few days were used to explore different folds (mostly thirds) and combinations of folds, as students examined which fractional amounts could and could not be easily produced.

What we were doing over these lessons was observing regularities, studying number patterns, learning about primes and composites, representing fractional amounts, making equivalent fractions, and practicing basic operations (mostly multiplication). Few, if any, of the students actually saw things in terms of formal concept development, however, with more than one asking some variation of, When are we going to have to do math again?

The query was repeated a few times during the lesson in which Fraction Kits were introduced, about a week into the unit. Consisting of red wholes, orange halves, yellow fourths, green eighths, and blue sixteenths — all made from neon-colored paper — these kits were intended to support the development of addition concepts by allowing students to compare and combine different-sized pieces more readily.

Kim was a bit disappointed when she opened her envelope. "There aren't any purple pieces in here," she protested.

I responded pragmatically: "There are only five different kinds of pieces, so I only used five colors. I was done before I got to the purple paper."

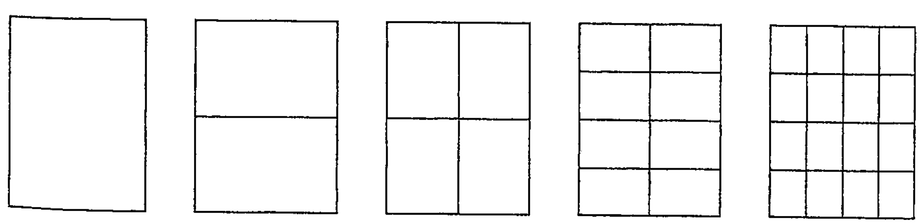
"You could add a different kind of piece," Kim offered.

"I think that would just get too confusing," I answered, not wanting to expand the kits before we'd had a chance to examine some of the relationships that they were intended to illustrate.

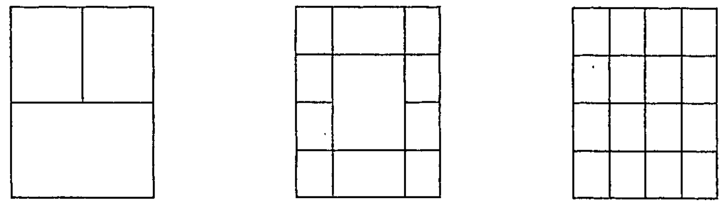
Kim acquiesced. Or, so I thought. We were well into an orienting activity of identifying some of the relationships among the pieces (framed in terms of "trading" certain parts for other parts) when Kim came up to me and asked if it would be all right if she were to make her own purple thirty-seconds. I agreed, and promised a sheet of neon purple if she reminded me at the end of the school day. She didn't forget.

In the next few lessons, we continued to explore the relationships among the pieces, focusing mainly around questions that asked students to find different ways of assembling a particular amount (e.g., What are some of the ways you can cover one fourth of a page?), of comparing amounts (e.g., Which is more, 3/4 or 13/16?), and of combining pieces (e.g., What do you get if you put together 2/4 and 1/16 and 1/8?). In every instance, I started the explorations with a few sample questions and then, upon a brief discussion of how they might be answered, invited students to make up their own questions.

It was these sorts of activities that led up to the "What are some of the ways you can use your Fraction Kits to cover a whole page?" task.



The **Fraction Kit**, consisting of wholes, halves, fourths, eighths, and sixteenths.



$1/2 + 1/4 + 1/4 = 1$ $1/4 + 8/16 + 2/8 = 1$ $16 \times 1/16 = 1$

Some other ways of covering a whole page using the Fraction Kits.

And, given the details about the unit that have been recounted here, it's not at all surprising that Kim would have given the response that was presented at the start of this chapter (that is, the product of an endless process of successive halving).

What may be surprising, though, is that she was the only one in the class of 25 who suggested the possibility. Her own groupmates, in fact, were only mildly interested in her drawing. As did most of their classmates, they chose to approach the task by reorganizing and trading pieces, making what appeared to be more or less random arrangements of pieces that covered a whole page. (A few examples are illustrated.) On my prompt, they also wrote addition and multiplication statements to describe and record what they'd done.

Other students didn't use the kits at all, choosing instead to list addition statements (e.g., $1/2 + 1/2 = 1$; $3/4 + 2/8 = 1$). One learner, Alex, began this way, but quickly realized there was a more efficient strategy for recording these combinations. In fact, he realized that he could make use of a table that was not only faster, but that could be used to generate *every* possible combination of pieces.

Two students working with Alex followed his lead in using the chart, but in very different ways. Jake, who couldn't quite follow what Alex was doing, made a chart in which he listed all the combinations in which the numerators and denominators were equal, proudly reporting to me that he knew "everything about one" after filling in the numbers along a diagonal. Tory, who also had trouble following Alex's logic, used the chart as Alex had set it up, but in a less systematic way (less as a generative tool and more as a recording tool).

Prompted by these events, I wondered what might happen if this strategy were introduced to the rest of the class. After calling for their attention, I presented my version of Alex's idea, being careful to represent the chart only as a recording tool. (As Jake's and Tory's responses had demonstrated, Alex's more abstract use to the charts to generate all possible combinations was not an easy jump to make.) I had in mind the hope that students' use of the charts might support more sophisticated understandings of the relationships among fractional amounts.

1	1/2	1/4	1/8	1/16
1				
	2			
	1	2		
	1	1	2	
	1	1	1	2
	1	1		4
	1		4	
	1		3	1
	1		2	4
	1		1	6
	1			8
		4		
		3	2	
		3	1	2
		3		4
		2	4	
		2	3	2
		2	2	4
		2	1	6
		2		8
		1	6	
		1	5	2
		1	4	4
		1	3	6
		1	2	8
		1	1	10
		1		12
			8	
			7	2
			6	4
			5	6
			4	8
			3	10
			2	12
			1	14
				16

Alex's Chart (Each row represents a different possibility. For example, in the circled row, the combination of 2/4 + 8/16 is recorded.)

The idea was taken up in earnest by every group. A sort of friendly competition quickly emerged as a set of students posed for themselves the challenge of generating a longer list than Alex's. Unhappy with the challenge, Alex soon convinced them that he had already generated the complete list — until a member of the second group, Lynn, suggested that it might be possible to use subtraction as well. (The topic of negative numbers had not been formally addressed, and in fact was several years away in this jurisdiction's mathematics curriculum.) Several new possibilities were quickly generated before Lynn added, "Hey, we can use parts of pieces too!" — noting, for example, that a combination of 3 fourth-pieces and a half of a half-piece covered the whole page.

Over the course of this 50-minute block, then, these 8- and 9-year-olds were adding, subtracting, multiplying, and dividing fractions, although my original intention with this activity was only that they have a little practice with some basic additive relationships.

As their teacher, I was quite excited about these events. Not only were these students demonstrating sound understandings of fraction concepts, they were showing that their understandings extended beyond the conceptual constraints of the kits. As such, the kits had served their purpose well. They'd provided a starting place, a location to develop a set of shared experiences that enabled learners to talk about and to extend their understandings of fractions.

Our fractions units lasted 1 more week, culminating in a "Fraction

1	1/2	1/4	1/8	1/16
1				
	2			
		4		
			8	
				16

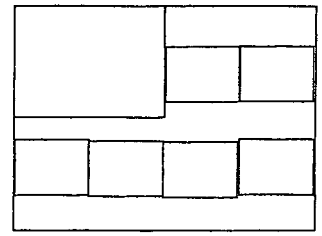
Jake's Chart

1	1/2	1/4	1/8	1/16
1				
	2			
		4		
	1	2		
		2		8
	1	1	1	2
		3	1	2

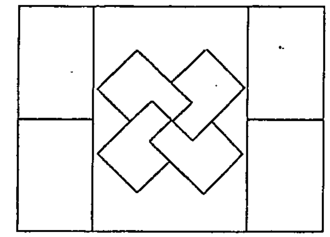
Part of Tory's Chart

1	1/2	1/4	1/8	1/16
1				
	2			
		4		
	3	−2		
2	−2			
2	−1	−2		
	1/2	3		

Some Entries in the Chart from Lynn's Group



Sofi's Flag (Her question was, How much of the background is still showing?)



Finn's Flag (His question was, How big is the combined piece in the middle?)

Flags" activity. Working from the kits, students were invited to create flag-like arrangements and then to pose questions based on those flags for themselves, their classmates, and their parents. A few examples have been re-presented here, but it's impossible to provide much of a sense of the richness, diversity, and conceptual sophistication of the work done by the students.

Perhaps a better sense might be conveyed through one parent's comment upon observing the posters that the children had made of their flags, their questions, and (hidden under a flap of paper) their responses: "I have no idea how to answer most of these. How did you teach the kids to do this?"

"You had to be here," I answered.