6 ALGORITHMS VERSUS NUMBER SENSE

We are usually convinced more easily by reasons we have found ourselves than by those which have occurred to others.

—*Blaise Pascal*

There still remain three studies suitable for free men. Arithmetic is one of them.

—*Plato*

Try an experiment. Calculate $\frac{6}{16} \times \frac{8}{18}$. Don't read on until you have an answer.

If you are like most people who are a product of the American school system, you probably got a pencil and paper, wrote the numbers down, and performed the following algorithm for multiplication of fractions. First you multiplied the numerators to get forty-eight. Then you multiplied the denominators (rewriting the multiplication vertically and performing the multiplication algorithm for whole numbers) to get 288. These actions resulted in the fraction $\frac{48}{288}$, which you then reduced to $\frac{1}{6}$ (perhaps even using several steps here). To check yourself, you probably went back and repeated the same actions and calculations; if you got the same answer twice, you assumed your calculations were correct.

Now take out a piece of graph paper and draw a rectangle. Use this rectangle to show the multiplication that represents the problem and what you did. See if you can find the rectangular arrays that represent the problems you did as you calculated the forty-eight and the 288, and then show in this rectangle the equivalence involved in reducing this fraction to $\frac{1}{6}$. If this is difficult for you, the way the algorithm was taught to you has worked against your own conceptual understanding of multiplication.

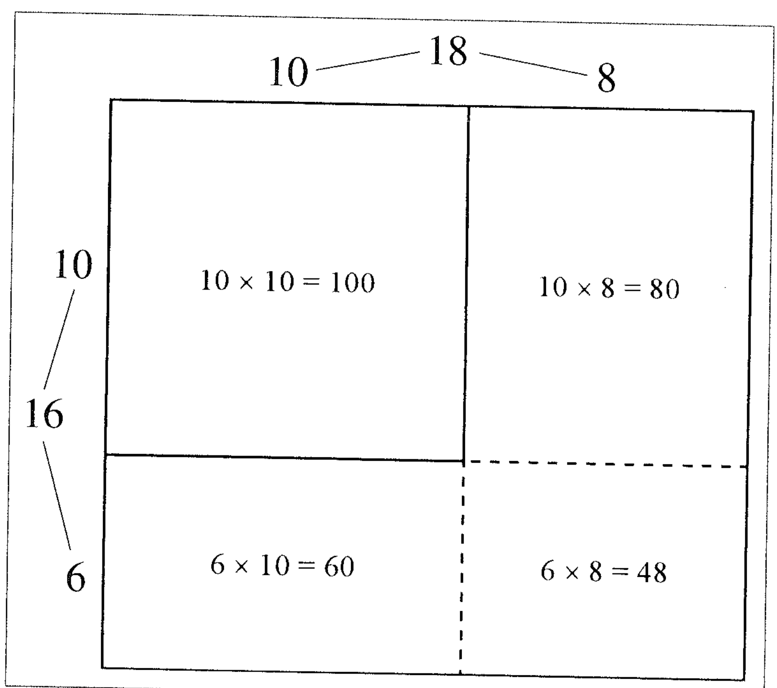
The algorithms for multiplying fractions are very difficult for children to understand. Why? Well, just think how nonsensical these steps must seem to them. They are struggling to understand what fractions even mean. Furthermore, they have often been taught—and therefore understand—the operation of multiplication as repeated addition, and they struggle to find the repeated addition when multiplying fractions. Finally, as they treat the numbers in the numerators and denominators as digits to perform the algorithm, they lose sight of the quantities they are actually multiplying and make any number of errors in calculating each of the separate pieces.

Liping Ma (1999) compared the way Chinese and American teachers think about and teach the multiplication algorithms and how they work with children who make place value mistakes. Most Chinese teachers approach the teaching of the multiplication algorithms conceptually.

For the whole number algorithm, they explain the distributive property and break the problem up into the component problems: $16 \times 18 = (10 + 6) \times (10 + 8) = (6 \times 8) + (6 \times 10) + (10 \times 8) + (10 \times 10) = 48 + 60 + 80 + 100$. Once this conceptual understanding is developed, they associate the steps in the algorithm with the component parts in the equation. (Figure 6.1 shows these steps as rectangles within the larger array of 16×18.) In contrast, 70 percent of American teachers teach this algorithm as a series of procedures and interpret children's errors as a problem with carrying and lining up. They remind children of the "rules"—that they are multiplying by tens and therefore have to move their answer to the next column. To help children follow the "rules" correctly, they often use lined paper and suggest that children use zero as a placeholder.

To teach the multiplication algorithm for fractions, Chinese teachers again approach it conceptually, focusing on both the distributive and the associative properties. They might explain that $\frac{6}{16}$ is equivalent to $6 \times \frac{1}{16}$ and $\frac{8}{18}$ is equivalent to $8 \times \frac{1}{18}$ and that therefore $\frac{6}{16} \times \frac{8}{18}$ is equivalent to $6 \times 8 \times \frac{1}{16} \times \frac{1}{18}$, or $48 \times \frac{1}{288}$. In contrast, American teachers are more likely to teach it procedurally, with rules like multiply the numerators, then the denominators, then reduce.

FIGURE 6.1
16 × 18 = 288 Components of the Algorithm



One could argue that if we taught the algorithms conceptually, as Liping Ma advocates, more understanding would develop. This is probably true. But should the algorithm be the goal of computation instruction? In today's world, do we want learners to have to rely on paper and pencil? Is the algorithm the fastest, most efficient way to compute? When are algorithms helpful? When does one pull out a calculator? What does it mean to compute with number sense?

Ann Dowker (1992) asked forty-four mathematicians to do several typical multiplication and division computation problems and assessed their strategies. Only 4 percent of the responses, across all the problems and across all the mathematicians, were solved with algorithms. The mathematicians looked at the numbers first, then found efficient strategies that fit well with the numbers. They made the numbers friendly, and they played with relationships. Interestingly, they also varied their strategies, sometimes using different strategies for the same problems when they were asked about them on different days! They appeared to pick a strategy that seemed appropriate to the numbers and that was prevalent in their minds at that time; they searched for efficiency and elegance of solution; they made numbers friendly (often by using landmark numbers); and they found the process creative and enjoyable.

How might mathematicians solve $\frac{6}{16} \times \frac{8}{18}$? There are many ways. One could, for example, swap the numerators. This makes the problem $\frac{8}{16} \times \frac{6}{18}$, or reduced to $\frac{1}{2} \times \frac{1}{3}$: the answer this way can be arrived at mentally. Why does this work? What does it mean to multiply $\frac{6}{16}$ by $\frac{8}{18}$? Figure 6.2 shows

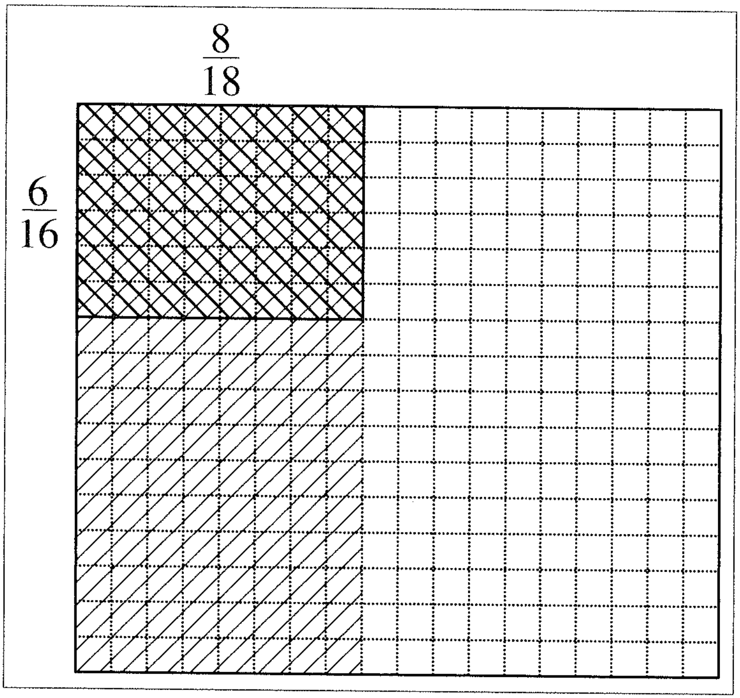


FIGURE 6.2
$\frac{6}{16} \times \frac{8}{18}$
A patio, 16 feet by 18 feet. $\frac{8}{18}$ of the tiles have been laid so far and $\frac{6}{16}$ of these have been mortared in place.

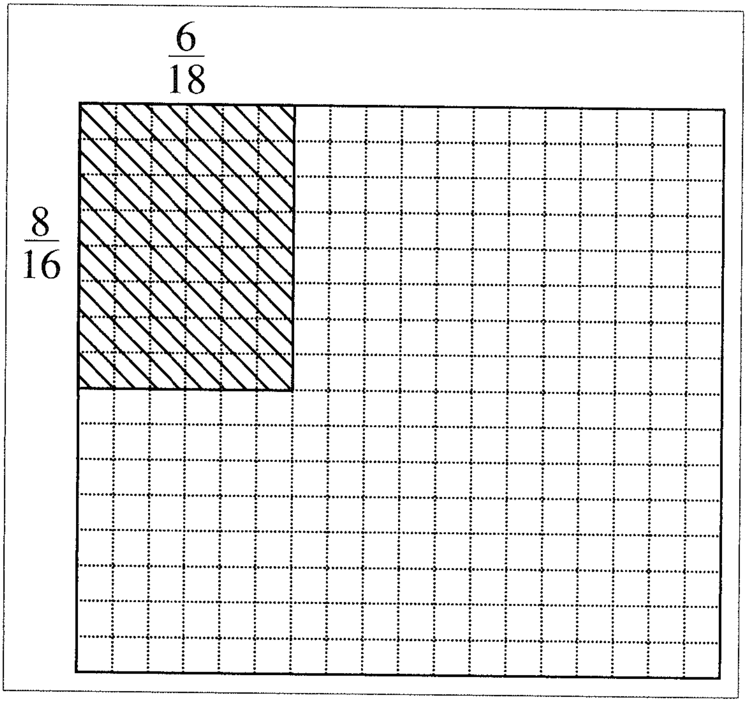
this multiplication in a rectangular array. Imagine square tiles being laid to build a patio, 16 feet by 18 feet. Next, imagine that $\frac{8}{18}$ of the tiles (8 out of 18 columns) have been laid so far and that $\frac{6}{16}$ of *these* (6 out of 16 rows) have been mortared in place. The small rectangular area that is now complete is a 6-by-8 array (48 tiles). This small array fits into the larger 16-by-18 array (288 tiles) six times. Thus, $\frac{6}{16} \times \frac{8}{18} = \frac{1}{6}$.

Now look at this in a different way. Imagine the smaller array turned 90 degrees: it now has 6 columns and 8 rows (see Figure 6.3). The relationship to the whole is still the same, but the problem is now $\frac{8}{16} \times \frac{6}{18}$, or $\frac{1}{2} \times \frac{1}{3}$, which of course is easily calculated mentally. Swapping numerators and reducing if needed is a powerful mental math strategy that is often helpful. For example, try it with $\frac{4}{6} \times \frac{3}{7}$, or $\frac{4}{5} \times \frac{5}{8}$. The first problem becomes $\frac{1}{2} \times \frac{4}{7}$, or $\frac{2}{7}$. The second becomes $\frac{5}{5} \times \frac{4}{8}$, or $\frac{1}{2}$. And it is easy to see in arrays how the smaller rectangular array, formed by the numerators, just gets turned: $(4 \times \frac{1}{7}) \times (3 \times \frac{1}{6})$ as $(4 \times 3) \times (\frac{1}{6} \times \frac{1}{7})$ or as $(4 \times \frac{1}{7}) \times (3 \times \frac{1}{6})$. See Figures 6.4a–6.4c.

This strategy is of course only helpful in some cases. But there are many wonderful mental math strategies if one has a deep understanding of number and operation. Calculating with number sense, as a mathematician, means having many strategies at your disposal, and looking to the numbers first, *before* choosing a strategy. Let's look at a few other strategies.

How about getting rid of fractions altogether? For $3\frac{1}{2} \times 14$, we could double the $3\frac{1}{2}$ to get rid of the fraction and thus halve the 14—turning the

FIGURE 6.3
$\frac{8}{16} \times \frac{6}{18}$



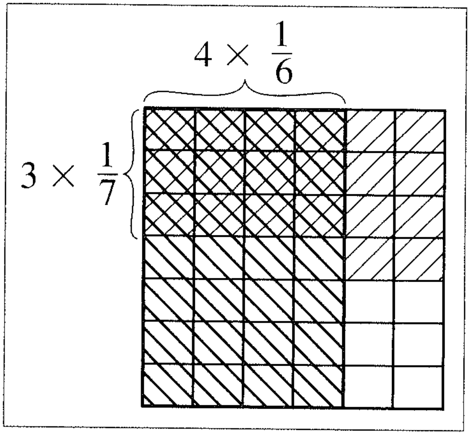


FIGURE 6.4a
$(3 \times 1/7) \times (4 \times 1/6)$

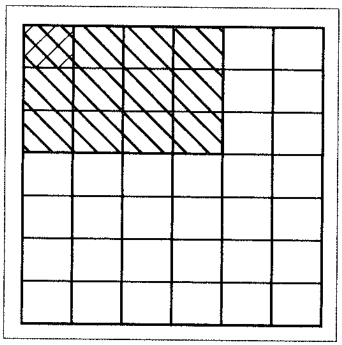


FIGURE 6.4b
$(3 \times 4) \times (1/6 \times 1/7)$

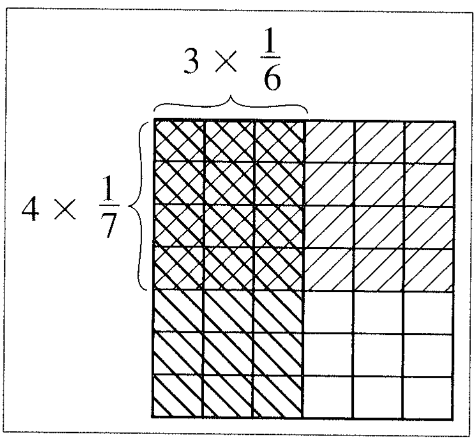


FIGURE 6.4c
$(4 \times 1/7) \times (3 \times 1/6)$

problem into 7 × 7! Now we have the answer of 49 mentally. For 2¼ × 16, we could get rid of the fraction by multiplying 2¼ by 4 and dividing the 16 by 4. This turns the problem into 9 × 4, or 36. Try using this strategy to compute 3⅕ × 45. Did you get rid of the fraction by multiplying 3⅕ by 5? Great! And then you divided 45 by 5? Great! Now you have 16 × 9. We could keep on doubling and halving. 16 × 9 = 8 × 18 = 4 × 36 = 2 × 72 = 144! Or, since we know that 16 × 10 = 160, all we have to do is subtract the extra 16 to get the answer of 144.

We can also use this strategy to get rid of decimals. How about .8 × 350? If we multiply the .8 by 10 and divide the 350 by 10, we turn the problem into 8 × 35. Halving and doubling, we get 4 × 70: 280! Or we could think of .8 as ⅘, turning the decimal into a fraction; ⅕ of 350 is 70; once again we get 4 × 70. All of these strategies work because of the associative property of multiplication. We can do whatever we want first to make the problems easier:

$$.8 \times 350 = ?$$

$$.8 \times 350 = 4 \times 70 = 280$$

$$.8 \times 350 = (8 \times \tfrac{1}{10}) \times (10 \times 35)$$
$$= 8 \times (\tfrac{1}{10} \times 10) \times 35 = \times 4 \times 2 \times 35$$

$$.8 \times 350 = \tfrac{4}{5} \times 350 = (4 \times \tfrac{1}{5}) \times 350 = 4 \times (\tfrac{1}{5} \times 350)$$

Note how all of these alternative, creative ways can be done so quickly—in most cases mentally. If paper and pencil are used, it is only to keep track. Playing with numbers like this is based on a deep understanding of number, landmark numbers, properties, and operations. And it characterizes true number sense. In contrast, a child who is taught to use the algorithm to multiply .8 × 350 stops thinking. He sacrifices the relationships in order to treat the numbers as digits. And any teacher of middle school children will attest to the difficulties children have as they try to complete each of the multiplication pieces, carry appropriately, and determine where the decimal point goes in the answer.

Algorithms can be very helpful when multiplying or dividing large, nonfriendly numbers, or when working with messy fractions that can't easily be simplified. But in today's world, isn't that when we take out the calculator anyway? If we have to reach for paper and pencil to perform the arithmetic, why not reach for the calculator?

THE HISTORY OF ALGORITHMS

Through time and across cultures many different algorithms have been used for multiplication. For example, for many years Egyptians used an algorithm based on doubling. To multiply 28 × 12 (or 12 × 28), they would calculate

$1 \times 28 = 28$, $2 \times 28 = 56$, $4 \times 28 = 112$, $8 \times 28 = 224$, and so on. As soon as they had calculations for numbers that added up to the original multiplier (in this case 12), they would stop doubling and add: here, $8 + 4 = 12$, therefore 112 and 224 added together (336) equals 12×28.

Russian peasants used a halving and doubling algorithm. To multiply 28×12, they would first halve the 28 and double the 12, getting 14×24. Next they would repeat the procedure, getting 7×48. When an odd number (like 7 in this case) appeared, resulting in a remainder when halved, they would round down, thus using 3 (instead of $3\frac{1}{2}$) $\times$ 96 and 1 (instead of $1\frac{1}{2}$) $\times$ 192. They continued halving and doubling until they reached the last problem in the series ($1 \times n$), in this case 1×192. Then they added up all the factors with odd multipliers—$48 + 96 + 192$ in this case—and arrived at the answer—$28 \times 12 = 336$.

In the early part of the ninth century, the great Arab mathematician Muhammad ibn Musa al-Khwarizmi invented the algorithms for multiplication and division that we teach in most schools today. (In Latin his name was Algorismus—hence the term *algorithm*.) Their beauty was that they were generalized procedures that could be used as efficient computation strategies for all problems—even messy ones with many digits. During this time, calculations using large numbers were needed both in the marketplace and for merchants' accounting purposes. Because calculations on the abacus were actions, there was no written record of the arithmetic, only the answer. And only the intelligentsia, practiced in the art of the abacus, could calculate.

Denis Guedj (1996) describes a bit of the history:

> In the Middle Ages computations were carried out on an abacus, also called a computing table, a calculating device resembling a table with columns or ruled horizontal lines; digits were represented by counters, or apices. From the twelfth century on, this type of abacus was progressively replaced by the dust board as a tool of calculations. This development did not come about without a struggle between those who, evoking the ancient Greek mathematician Pythagoras, championed the abacus and those who became masters of algorism, the new Arabic number system. In this competition between the Ancients and Moderns, the former saw themselves as the keepers of the secrets of the art of computation and the defenders of the privileges of the guild of professional calculators . . . [while] the new system indisputably marked the democratization of computation. (53–54)

With the invention of the algorithms and the dissemination of multiplication tables to use while performing them, even the most complex computations were possible, and written records of the calculations could be kept.

Schools soon set about to teach the procedures. In the Renaissance in Europe the manipulation of numbers and the practice of arithmetic were signs of advanced learning; those who knew how to multiply and divide with algorithms were guaranteed a professional career. In the Musee de Cluny, in Paris, there is a sixteenth-century tapestry depicting Lady Arithmetic teaching the new calculation methods to gilded youth. (A photograph of this is used as the lead photo in this chapter.)

But today's world is different. Human beings have continued through the centuries to design and build tools with which to calculate, from the slide rule, in 1621, to the first mechanical calculator, invented by Pascal in 1642, to the handheld calculator, in 1967, to today's graphic calculators. The World Wide Web even provides virtual calculators (Guedj 1996). Difficult computations, originally solved by algorithms, are now done with these tools.

There have also been many different algorithms for computation with fractions. As described in Chapter 3, in the Stone Age there was no need for fractions; fractions seem to have developed during the Bronze Age. Egyptians during this period recognized only unit fractions (fractions with numerators of one) and the fraction $\frac{2}{3}$. Thus they would have understood the fractions $\frac{6}{16}$ or $\frac{8}{18}$ only as six loaves shared with sixteen people and eight loaves shared with eighteen people, representing $\frac{6}{16}$ as $\frac{1}{3} + \frac{1}{24}$ and $\frac{8}{18}$ as $\frac{1}{3} + \frac{1}{9}$. Adding or subtracting these amounts is easy. No common denominators are needed. You just string all the unit fractions together: $\frac{1}{3} + \frac{1}{24} + \frac{1}{3} + \frac{1}{9} = \frac{2}{3} + \frac{1}{9} + \frac{1}{24}$.

Fractions were multiplied by doubling (or tripling) one of the denominators and divided by doubling (or tripling) one of the numerators. For example, to multiply $\frac{1}{2} \times \frac{1}{3}$, one would just double the 3 or triple the 2 to get an answer of $\frac{1}{6}$. To multiply $\frac{1}{4} \times \frac{1}{3}$, one would double the 3 twice since $\frac{1}{4}$ is half of $\frac{1}{2}$—the 3 becomes 6, then the 6 becomes 12. Thus the answer is $\frac{1}{12}$. To divide $\frac{1}{3}$ by $\frac{1}{2}$, one would double the numerator. Thus the answer is $\frac{2}{3}$. To multiply $\frac{6}{16} \times \frac{8}{18}$, the problem at the beginning of this chapter, Egyptians would have used the distributive property and done the following: $(\frac{1}{3} + \frac{1}{24}) \times (\frac{1}{3} + \frac{1}{9}) = \frac{1}{9} + \frac{1}{27} + \frac{1}{72} + \frac{1}{216}$. Imagine how difficult it would be to do very complex problems with the Egyptian algorithm!

Had Mesopotamian mathematics, like that of the Nile Valley, been based on the addition of integers and unit fractions, we might not have seen decimal calculations until the Renaissance! However, their neighbors, the Babylonians (as also described in Chapter 3), had developed a base-*sixty* number system. They used this to represent fractional amounts, *sexagesimals*. Knowledge of how place value could be used allowed the Babylonians to do all their calculating of fractions in the same way as they did whole numbers, taking care of the decimal (really, sexagesimal) point only at the end.

The handheld calculator has now replaced paper-and-pencil algorithms. Does this mean we don't need to know how to calculate? Of course not. To be successful in today's world, we need a deep conceptual understanding of mathematics. We are bombarded with numbers, statistics, advertisements, and similar data every day—on the radio, on television, and in newspapers. We need good mental ability and good number sense in order to evaluate advertising claims, estimate quantities, efficiently calculate the numbers we deal with every day and judge whether these calculations are reasonable, add up restaurant checks and determine equal shares, interpret data and statistics, and so on. We need to be able to move back and forth from fractions to decimals to percents. We need a much deeper sense of number and operation than ever before—one that allows us to both estimate and make exact calculations mentally. How do we, as teachers, develop children's ability to do this? How do we engage them in being young mathematicians at work?

TEACHING FOR NUMBER SENSE

Each day at the start of math workshop, Dawn Selnes, a fifth-grade teacher in New York City, does a short minilesson on computation strategies. She usually chooses five or six related problems and asks the children to solve them and share their strategies with one another. Crucial to her choice of problems is the relationship between them. She picks problems that are likely to lead to a discussion of a specific strategy. She allows her students to construct their own strategies by decomposing numbers in ways that make sense to them. Posted around the room are signs the children have made throughout the year as they have developed a repertoire of strategies for operations with fractions. One reads, "Make use of tens"; another, "Halves & doubles"; a third, "Get rid of the fraction"; a fourth, "Use all the factors in pretty ways."

On the chalkboard today is the string of problems the children are discussing. Although the string ends with fractions, it begins with a few whole number multiplication problems, and Alice is describing how she solved 9 × 30. "I just used all the factors," she explains. "I thought of it as nine times three times ten. I knew that nine times three was twenty-seven, so times ten is two hundred and seventy."

Dawn asks for other strategies, but most of the children have treated the problem similarly, so Dawn goes to the next problem in her string, 15 × 18. Several children use the distributive property here. Tom's strategy is representative of many, and several children nod in agreement as he explains how he did 10 × 18 and got 180, and then took half of that to figure out the answer to 5 × 18. He completes the calculation by adding 180 to 90, for an answer of 270. Lara's strategy is similar, if not as elegant, but it makes sense to her. She multiplies using tens, too, but she breaks up the eighteen instead of the fifteen and multiplies 10 × 15, and then 8 × 15. These two products together also result in 270.

Ned agrees with their answer but with a smile he says, "Yeah, but you didn't even have to calculate. It's the same as nine times thirty, because the thirty is halved, and the nine is doubled!"

Although all the children in the class are comfortable with this doubling and halving strategy and understand why it works (having explored it thoroughly with arrays earlier in the year), they have not all thought to use it, because Dawn has turned the numbers around. It might have been more obvious if she had written 18 × 15 directly underneath 9 × 30. But she wants to challenge them to think.

Now Dawn moves to fractions. She writes 4½ × 60 as the third problem. Several children immediately raise their hands, but Dawn waits for those still working to finish. Alice is one of them, so she asks her to share first. "What did you do, Alice?"

"I split it into four times sixty first," Alice begins, "and I did that by doing four times six equals twenty-four. Then times ten is two hundred and forty. Then I knew that a half of sixty was thirty. So thirty plus two hundred and forty is two hundred and seventy."

"My way is kind of like yours," another classmate, Daniel, responds, "but I subtracted."

"But then you would get the wrong answer," Alice tells him, looking puzzled.

"No, what I mean is I did five times six times ten. That was three hundred. Then I subtracted the thirty."

"Where did you get the five?" Several of his classmates are also now puzzled.

"That was easier for me than four and a half. But that's why I took thirty away at the end," Daniel explains, very proud of his strategy.

Dawn checks to see whether everyone understands by asking who can paraphrase Daniel's strategy. Several children do so, and Dawn seems satisfied that the group appears to understand. "That's a really neat strategy, isn't it?" Daniel beams, and Dawn turns to Ned, "And what did you do Ned? Your hand was up so quickly. Did you see a relationship to another problem again?"

Ned laughs, "Yep. Just doubling and halving again. It's the same as nine times thirty. The nine was halved and the thirty was doubled."

Several children make surprised exclamations. Dawn smiles and goes to the next problem: 2¼ × 120. This time everyone's hand is up quickly, and Dawn calls on Tanya, who has not yet shared. Tanya, as well as the rest of the class, has made use of the doubling and halving relationships in this string of problems.

The other strategies that have previously been offered are also powerful strategies, and Dawn does not want to imply that they should be replaced by doubling and halving. She is only trying to help her children think about relationships in problems, to look to the problems *first* before calculating. To ensure that this happens she follows with the next two problems: 15 × 36, then 15½ × 36. For the first, most students see the relationship between it and 30 × 18. Since they have already calculated 15 × 18, they know they just need to double that answer. A few children solve it by doing 10 × 36 to get 360, halving that to get 180, and then adding these partial products for an answer of 540. For the second problem everyone uses the distributive property, adding 18 more for an answer of 558.

Dawn ends her string with a very difficult problem: $15\frac{1}{2} \times 4\frac{1}{2}$. She asks the children to write their strategy and solution down in their math journal and then to turn to the person sitting next to them on the rug and share it. What strategies will the children use? How solid is their understanding? Most complete the problem successfully, but not all children finish, and some make calculation errors. But they show a rich variety of strategies that are evidence of deep understanding and good number sense (see Figure 6.5).

These young mathematicians are composing and decomposing flexibly as they multiply fractions. They are inventing their own strategies. They are looking for relationships between the problems. They are looking at the numbers first before they decide on a strategy.

Children don't do this automatically. Dawn has developed this ability in her students by focusing on computation during minilessons with strings of related problems every day. She has developed the big ideas and models through investigations, but once this understanding has been constructed, she hones computation strategies in minilessons such as this one.

Traditionally, mathematics educators thought teaching for number sense meant helping children connect their actions to real objects. We used

$$15\frac{1}{2} \times 4\frac{1}{2}$$

$$270 \div 4 = 135 \div 2 =$$

$$4\frac{1}{2} \times 60 = 270 \quad \div 4$$

$$4\frac{1}{2} \times 15 \overset{\div 4}{=} 67\frac{1}{2}$$

$$4\frac{1}{2} \times 15\frac{1}{2} = 67\frac{1}{2} + 2\frac{1}{4} = 69\frac{3}{4}$$

$$15 \times 4\frac{1}{2} = \qquad 15 \times 36 = 540$$

$$15\frac{1}{2} \times 36 = 558 \qquad \frac{1}{2} \times 36 = 18$$

$$15\frac{1}{2} \times 4\frac{1}{2} = 558 \div 8 = 69\frac{3}{4}$$

$$9 \times 30 = 270$$

$$4\frac{1}{2} \times 15 = 67\frac{1}{2}$$

$$\frac{1}{2} \times 4\frac{1}{2} = 2\frac{1}{4}$$

$$69\frac{3}{4}$$

FIGURE 6.5 *$15\frac{1}{2} \times 4\frac{1}{2}$: Three Different Children's Strategies* base-ten blocks and trading activities to help children understand regrouping. We built arrays with base-ten materials and looked at the dimensions and the area. We used Cuisennaire rods and fraction strips to develop a connection for children between the actions of regrouping the objects, making equivalent fractions, and the symbolic notation in the algorithms. We talked about the connection between the concrete, the pictorial, and the symbolic. But all of these pedagogical techniques were used to teach the algorithms. The goal of arithmetic teaching was algorithms, albeit with understanding.

In the 1980s, educators began to discuss whether the goal of arithmetic computation should be algorithms at all. Constance Kamii's research has led her to insist that teaching algorithms is in fact harmful to children's mathematical development (Kamii and Dominick 1998). First, she examined children's invented procedures for whole number multiplication and division and found that children's procedures for multiplication always went from left to right, from the largest units to the smallest. With division, children's

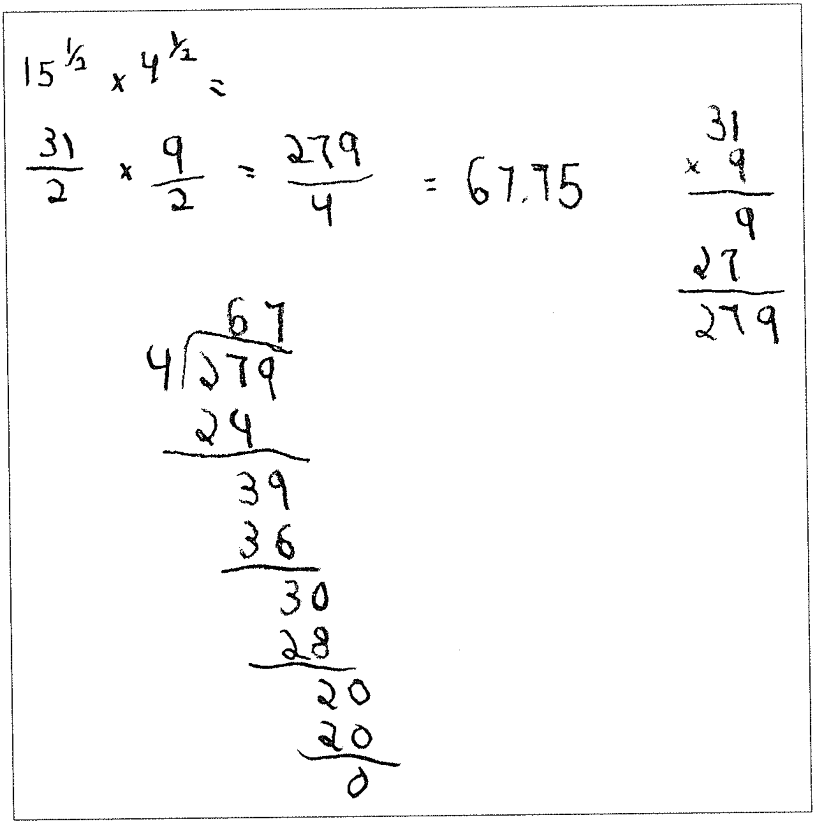


FIGURE 6.6 *15½ × 4½: One Child's Attempt at the Algorithm*

procedures went from the smallest units to the largest, from right to left. Yet the algorithms require opposite procedures: with multiplication one starts with the units and works right to left; with division, one starts with the largest unit (hundreds, for example) and works right to left.

Is the situation any different with fractions? Only one child in Dawn's class used the algorithm, and he had been taught it at home. He turned the problem into $3\frac{1}{2} \times \frac{9}{2} = \frac{279}{4}$ (see Figure 6.6). This procedure was obviously difficult for him, and he made many errors along the way. We might also wonder if he knows why this procedure works. When algorithms are taught as procedures to use for any and all problems, children necessarily give up their own meaning making in order to perform them. The algorithms hinder children's ability to construct an understanding of the distributive and associative properties of multiplication, which underlie algebraic computation. And worse, they require that children see themselves as proficient users of someone else's mathematics, not as mathematicians.

Kamii's data and her strong arguments from a developmental perspective are convincing, and many educators have begun to allow children to construct their own computation strategies. This isn't enough, however. Although their invented strategies do become more efficient over time, these strategies are remarkably similar, and many of them are cumbersome and inefficient.

Over the last seven years or so Mathematics in the City has looked seriously at how to develop in students a repertoire of efficient computation strategies that are based on a deep understanding of number sense and operation and that honor children's own constructions. The next chapter describes the techniques we have been using and the strategies we try to develop for fraction and decimal computation.

SUMMING UP . . .

Algorithms were developed in the Middle Ages by the Arab mathematician al-Khwarizmi. There was also a long period when computations were performed with unit fractions and/or sexagesimals before common fraction algorithms became accepted. The use of algorithms brought about a democratization of computation; people no longer had to rely on the select few who were competent users of the abacus. When algorithms appeared, there was political tension between those who wanted to hold on to the abacus and those who wanted to learn the new methods. Interestingly, a similar political situation exists today. As schools have begun to reform their teaching, as algorithms have been replaced with mental math strategies and calculating with number sense, arguments have broken out between those who fight to maintain the "old" math and those who favor reform. Many newspaper articles play into the fear that children will not be able to compute. This fear is based on uninformed, often mistaken, notions of the reform. Parents are products of the old education, and therefore they define mathematics as the skills they were taught. When they don't see their children learning what they believe to be the goals of mathematics—the algorithms—they assume that nothing is being learned. Many of them have called the new mathematics "fuzzy" or "soft" and described it as a "dumbing down."

Algorithms—a structured series of procedures that can be used across problems, regardless of the numbers—do have an important place in mathematics. After students have a deep understanding of number relationships and operations and have developed a repertoire of computation strategies, they may find it interesting to investigate why the traditional computation algorithms work. Exploring strategies that can be used with larger, messy numbers when a calculator is not handy is an interesting inquiry—one in which the traditional algorithms can be employed. In these inquiries algorithms can surface as a formal, generalized procedure—an alternative approach to use when the numbers are not nice. Often algorithms come up in classroom discussions, too, because parents have taught them to their children and children attempt to use them without understanding why they work. Exploring them and figuring out why they work may deepen children's understanding.

Algorithms should not be the primary goal of computation instruction, however. Using algorithms, the same series of steps with all problems, is antithetical to calculating with number sense. Calculating with number sense means that one should look at the numbers first and then decide on a strategy that is fitting—and efficient. Developing number sense takes time; algorithms taught too early work against the development of good number sense. Children who learn to think, rather than to apply the same procedures by rote regardless of the numbers, will be empowered. They will not see mathematics as a dogmatic, dead discipline, but as a living, creative one. They will thrive on inventing their own rules, because these rules will serve afterward as the foundation for solving other problems.

By abandoning the rote teaching of algorithms, we are not asking children to learn less, we are asking them to learn more. We are asking them to mathematize, to think like mathematicians, to look at the numbers before they calculate. To paraphrase Plato, we are asking children to approach mathematics as "free men and women." Children can and do construct their own strategies, and when they are allowed to make sense of calculations in their own ways, they understand better. In the words of the mathematician, Blaise Pascal, "We are usually convinced more easily by reasons we have found ourselves than by those which have occurred to others."

In focusing on number sense, we are also asking teachers to think mathematically. We are asking them to develop their own mental math strategies in order to develop them in their students. Once again teachers are on the edge, not only the edge between the structure and development of mathematics, but also the edge between the old and the new—between the expectations of parents and the expectations of the new tests and the new curricula.

The backlash is strong, and walking this edge is difficult. Teachers need support. Learning to teach in a way that supports mathematizing—in a way that supports calculating with number sense—takes time. Sometimes, parents have responded by hiring tutors to teach their children the algorithms—a solution that has often been detrimental to children as they grapple to understand number and operation. Sometimes, as teachers have attempted to reform their practice, children have been left with no algorithms and no repertoire of strategies, only their own informal, inefficient inventions. The reform will fail if we do not approach calculation seriously, if we do not produce children who can calculate efficiently. Parents will define our success in terms of the their old notions of mathematics. They saw the goal of arithmetic, of school mathematics, as calculation. They will look for what they know, for what they learned, for what they define as mathematics.