2 THE LANDSCAPE OF LEARNING

It is not knowledge but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment. When I have clarified and exhausted a subject, then I turn away from it, in order to go into darkness again; the never satisfied man is so strange. . . . If he has completed a structure, then it is not in order to dwell in it peacefully, but in order to begin another. I imagine the world conqueror must feel thus, who, after one kingdom is scarcely conquered, stretches out his arms for others.

—*Karl Friedrich Gauss*

Mathematics is not a careful march down a well-cleared highway, but a journey into a strange wilderness, where the explorers often get lost. Rigour should be a signal to the historian that maps have been made, and the real explorers have gone elsewhere.

—*W. S. Anglin*

DESCRIBING THE JOURNEY

Linear Frameworks

Historically, curriculum designers did not use a developmental framework like Carol's when they designed texts, nor did they see mathematics as mathematizing—as activity. They employed a teaching/learning framework based on the accumulated content of the discipline. They analyzed the structure of mathematics and delineated teaching and learning objectives along a line. Skills were assumed to accumulate eventually into concepts (Gagné 1965; Bloom et al. 1971). For example, simplistic notions of fractions were considered developmentally appropriate for early childhood if they were taught as a shaded part of a whole or with pattern blocks. Later, around third grade, the equivalence of fractions was introduced, and still later, in fifth or sixth grade, operations with fractions. Development was considered but only in relation to the content: from simple to complex skills and concepts.

Focusing only on the structure of mathematics leads to a more traditional way of teaching—one in which the teacher pushes the children toward procedures or mathematical concepts because these are the goals. In a framework like this, learning is understood to move along a line. Each lesson, each day, is geared to a different objective, a different "it." All children are expected to understand the same "it," in the same way, at the end of the lesson. They are assumed to move along the same path; if there are individual differences it is just that some children move along the path more slowly—hence, some need more time or remediation. Figure 2.1 depicts such a linear framework.

Learning Trajectories

As the reform mandated by the National Council for Teachers of Mathematics has taken hold, curriculum designers and educators have tried to develop other frameworks. Most of these approaches are based on a better understanding of children's learning and of the development of tasks that will challenge them. One important finding is that children do not all think the same way. These differences in thinking are obvious in the dialogue in Carol's classroom. Although all the children in the class worked on the submarine sandwich problem, they worked in different ways, exhibited different strategies, and acted in the environment in different mathematical ways.

Marty Simon (1995) describes a learning/teaching framework that he calls a "hypothetical learning trajectory." The learning trajectory is hypothetical because, until students are really working on a problem, we can never be sure what they will do or whether and how they will construct new interpretations, ideas, and strategies. Teachers expect their students to solve a problem in a certain way. Or, even more refined, their expectations are different for different children. Figure 2.2 depicts a hypothetical learning trajectory.

Simon uses the metaphor of a sailing voyage to explain this learning trajectory:

> You may initially plan the whole journey or only part of it. You set out sailing according to your plan. However, you must constantly adjust because of the conditions that you encounter. You continue to acquire knowledge about sailing, about the current conditions, and about the areas that you wish to visit. You change your plans with respect to the order of your destinations. You modify the length and nature of your visits as a result of interactions with people along the way. You add destinations that prior to the trip were unknown to you. The path that you travel is your [actual] trajectory. The path that you anticipate at any point is your "hypothetical trajectory." (136–37)

As this quote makes clear, teaching is a planned activity. Carol did not walk into her classroom in the morning wondering what to do. She had planned her lesson, and she knew what she expected her students to do. As the children responded, she acknowledged the differences in their thinking and in their strategies, and she adjusted her course accordingly. While she honored divergence, development, and individual differences, she also had identified landmarks along the way that grew out of her knowledge of mathematics and mathematical development. These helped her plan, question, and decide what to do next.

Over the last five years, the Mathematics in the City staff have been helping teachers like Carol develop and understand what we originally called "learning lines"—hypothetical trajectories comprising the big ideas, the mathematical models, and the strategies that children construct along the way as they grapple with key mathematical topics (addition and subtraction; multiplication and division; fractions, decimals, and percents). In conjunction with these teachers, we analyzed children's work, we looked at videotapes of lessons, and we interviewed children. We discussed the *strategies* (and their progression—the schematizing) that children used as they acted within the environment mathematically. We attempted to specify the important *big ideas* the children grappled with for each topic. And we focused on *mathematical modeling,* whereby students see, organize, and interpret their world mathematically.

Although we still believe that knowledge of models, strategies, and big ideas will enable teachers to develop a "hypothetical learning trajectory," we have stopped calling it a learning line—the term seems too linear. Learning—real learning—is messy (Duckworth 1987). We prefer instead the metaphor of a landscape.

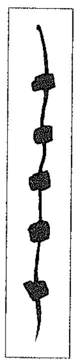

FIGURE 2.1
Linear Framework

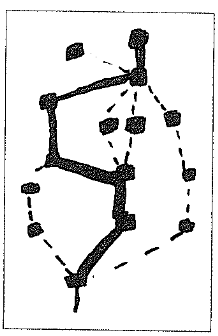

FIGURE 2.2
Hypothetical Learning Trajectory (Simon 1995)

The big ideas, strategies, and models are important landmarks for Carol as she journeys with her students across the landscape of learning. As she designs contexts for her students to explore, her goal is to enable them to act on and within the situations mathematically and to trigger discussions about the landmarks. Carol also has horizons in mind when she plans—horizons like understanding fractions as division and equivalence of fractions to decimals and percents. As she and the children move closer to a particular horizon, landmarks shift, new ones appear.

The paths to these landmarks and horizons are not necessarily linear, and there are many such paths, not just one. As in a real landscape, the paths twist and turn; they cross one another, are often indirect. Children do not construct each of these landmark ideas and strategies in an ordered sequence. They go off in many directions as they explore, struggle to understand, and make sense of their world mathematically. Strategies do not necessarily affect the development of big ideas, or vice versa. Often constructing a big idea, like fractions as division, will affect learners' strategies for finding equivalence; but just as often "trying out" new strategies for finding equivalent fractions and then investigating why they work may help students construct insightful relationships. Ultimately, what is important is how children function in a mathematical environment (Cobb 1997)—how they mathematize.

It is not up to us, as teachers, to decide which pathways our students will use. Often, to our surprise, children will use a path we have not encountered before. That challenges us to understand the child's thinking. What is important, though, is that we help all our students reach the horizon. When we drive a car down the road, our overall attention is on the horizon. But we also see the line in the middle of the road and use it to direct the car in the right direction. Once that line is behind us, however, it no longer serves that purpose. It is the same with teaching. When a child still needs to draw and cut wholes up into equal parts to determine equivalent fractions, the teacher designs activities to support the development of fair sharing. However, when a child understands how to use ratio tables and has a variety of strategies for arriving at equivalent fractions and decimals, when it seems those landmarks have been passed, the teacher has already shifted attention to more-distant landmarks on the horizon like operating with fractions.

When we are moving across a landscape toward a horizon, the horizon seems clear. But as we near it, new objects—new landmarks—come into view. So, too, with learning. One question seemingly answered raises others. Children seem to resolve one struggle only to grapple with another. Teachers must have the horizons in mind when they plan activities, when they interact, question, and facilitate discussions. But horizons are not fixed points in the landscape; they are constantly shifting. Figure 2.3 depicts the landscape-of-learning framework.

Thinking of teaching and learning as a landscape suggests a beautiful painting. But if learners can take so many paths and the horizons are constantly shifting, how do teachers ever manage? How do we help each child make the journey and still keep in mind the responsibility we have for the class as a whole?

Carol chooses a context (field trips and submarine sandwiches) and structures fair sharing and comparing within this context because she knows that understanding fractions as division—three subs shared equally among four kids results in $\frac{3}{4}$ of a sub—is a big idea. She chooses to discuss the equivalence of $\frac{1}{10}$ to $\frac{1}{5}$ of $\frac{1}{2}$ because she knows that understanding fractions requires understanding the relationship of parts to the whole, even when the whole changes. She is aware, as she walks around the room, of the strategies children are using—whether they draw and compare parts or use landmark fractions and their equivalents flexibly and mentally. She notices because she knows these strategies are significant in mathematical development—they represent the ways children are *schematizing* in a mathematical environment (Cobb 1997).

Word Problems vs. Truly Problematic Situations

One could argue that the use of context in mathematics teaching is not new. Certainly we all have vivid memories of word problems. Usually, however, our teachers assigned them after they had explained operations, algorithms (like invert and multiply), or rules for equivalent fractions, and we were expected to apply these procedures to the problems. In Carol's class, context is not being used for *application* at the end of a unit of instruction. It is being used at the start, for *construction*. Nor is the submarine sandwich context a trivial, camouflaged attempt to elicit "school mathematics." It is a rich, truly problematic situation that is real to the students, that allows them to generate and explore mathematical ideas, that can be entered at many levels, and that supports mathematizing.

Much reform is currently underway in schools in accordance with the new National Council for Teachers of Mathematics (NCTM) *Principles and Standards for School Mathematics* (2000), and many teachers are attempting to use problems to construct understanding rather than teach by telling. But many of the problems teachers introduce are still traditional word problems. Join us in another classroom, and we'll show you what we mean.

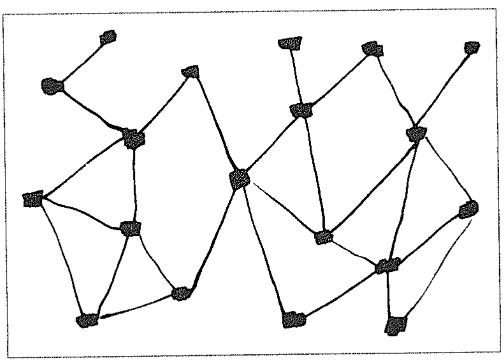

FIGURE 2.3
Landscape of Learning

Susan, a fifth-grade teacher, is reading to five children grouped around her. "'Maria went to the music store—'" She turns to the children. "Do you want me to read it, or do you want to?"

Several of the children chorus, "You."

Susan continues. "'Maria went to the music store to buy some CDs. She had twenty-four dollars with her. She spent eighteen dollars. What fractional part of her money did she spend?"' She pauses, thinks about what she has read, and then, although it is not part of the original problem, adds, "in its lowest form."

Susan begins by involving the children in solving a problem. She is not asking them to apply an algorithm for equivalent fractions, but instead she asks them to think—to solve the problem in a way that makes sense to them. She is attempting to promote construction, not application. She is clear about the mathematics (fraction as a part of a whole) she wants the children to explore, and she structures the context to support development, just as Carol did. But could Susan's context be stronger? Do the children become invested in the problem? Do they mathematize it? Or is it just a "school-type trivialized word problem"?

One of the children, Michael, starts to take the fraction bars out of a nearby bin, then puts them back, commenting, "Oh, I don't need these, easy." Other children comment that they are confused. One of these children, Josh, questions, "Just eighteen dollars? What do you mean—wasn't there tax? What do you mean lowest form?"

Susan attempts to clarify the confusion. "Look at my question. What fractional part of her money did she spend?" She pauses, then adds, "The eighteen dollars included the tax. So she spent eighteen of the twenty-four dollars. What fraction is that?"

This how-much-money scenario is one children can imagine, and in that sense it is realistic. But it is not likely to promote mathematizing. It is not likely to cause children to interpret their lived world on the basis of mathematical models. It is closed, with an expected answer of $^{18}/_{24}$ reduced to $^{3}/_{4}$—a camouflaged attempt at eliciting a fraction and reducing. Why would one ever wonder what fractional part of the twenty-four dollars was spent? In this situation, one usually only wonders how much money one has left. No wonder there is initial confusion when Susan reads the problem. In real life, there is tax, and there is no need to calculate a fraction, or to reduce. Because the children are confused, Susan must clarify. She attempts to steer the children toward the fraction that she wants them to make and reduce by rephrasing the question: "She spent eighteen of the twenty-four dollars. What fraction is that?" Unfortunately, now there is almost nothing left to solve. The context becomes irrelevant, and the children will sacrifice their own meaning making to accommodate what Susan wants.

Michael and Nora respond, "Oh, that's easy." Annie, however, remains confused.

Susan repeats the problem. "Think about it. Maria spends eighteen dollars out of the twenty-four she had. What fraction is that?"

Annie quickly says, "Oh, eighteen twenty-fourths—no."

At first Susan does not acknowledge the correctness of Annie's answer, responding, "Think about it, because you'll have to tell me how you figured it out, won't you?"

But Annie's confusion is still apparent. "I don't get it."

"Okay." Susan attempts to give Annie more time. "Josh is going on to think of another way to figure it out. Maybe the rest of you would like to find another way, too, while we give Annie more time to think about it."

Annie responds with more conviction, "Eighteen twenty-fourths—because the eighteen goes on the top and the twenty-four goes on the bottom."

This time Susan acknowledges Annie's thinking. "Okay, so now what is its lowest form? And think about how you will explain your work."

Susan is patient as she reminds Annie that she will have to explain her thinking. She does not supply an answer, nor does she acknowledge the correctness of Annie's first solution—that would stop her from thinking. To give Annie the time she needs, Susan encourages the other children to work on another strategy. But is the problem rich enough to benefit from exploring alternative strategies? What alternative strategies are there?

Teachers often confuse tools with strategies. Unifix cubes or fraction bars or paper and pencil are not different strategies. They are different tools. Representing the problem with stacks of Unifix cubes, or with fraction bars, or by drawing twenty-four dollars and circling eighteen of them are all the same mathematically. No benefit is derived by changing tools unless the new tool helps the child develop a higher level of schematizing (in this case, enables the child to construct the whole and the part and equate them to the reduced relation). Is this context rich enough for that?

Susan turns to all the children and invites them to begin a discussion. "Who would like to explain how he or she figured it out? And I would like the rest of you to listen, and if you have a question, ask."

Annie offers to begin. "If Maria had twenty-four dollars, then eighteen of that is eighteen twenty-fourths [*she counts the appropriate number of Unifix cubes as she explains*] . . . twenty-one, twenty-two, twenty-three, twenty-four. So these [*she points to a group of eighteen*] are eighteen twenty-fourths."

Susan points to the cubes and acknowledges Annie's statement. "This much is eighteen twenty-fourths of the whole." Then she turns to Michael. "Michael, you did it without manipulatives—you started to take them and then put them back. Can you explain what you did?"

"Yeah, you just take the number eighteen and put it as the numerator. The denominator is twenty-four. So I just knew it, eighteen twenty-fourths. Then you reduce. You divide each number by six and you get three fourths."

"So you just knew that the fraction eighteen twenty-fourths could be reduced to three fourths?" Susan rephrases. "Any different ways? Josh?"

Josh's strategy is similar to Annie's. He also counts, but he has drawn twenty-four dollars and circled eighteen. "There are eighteen out of twenty-four. The eighteen goes on top and the twenty-four on the bottom. Then I cut each number in half and got nine twelfths. Then I divided each by three and got three fourths."

Susan asks, "Did you just know how to reduce too, or did you use your drawing to prove it?" Josh acknowledges that he just knew. Susan then turns to Nora, who comments, "I just knew, too." Susan concludes the lesson with, "That's something that is really neat about our number system. You can reduce fractions by dividing the numerator and the denominator by the same number. It's like dividing the whole fraction by the number one. Michael, you used six sixths, and what is that equal to?"

Several children murmur, "One."

"Right. And, Josh you used two halves and then three thirds. These fractions equal one, too. Remembering that makes reducing to the lowest form easy."

Note the language the children use: "take the number eighteen and put it as the numerator," "the denominator is twenty-four," "put the eighteen on the top." They treat the problem abstractly. When a context is real and meaningful for children, their conversation relates to the context. They mathematize the situation. They talk about money, about fair sharing and portioning. There is a reason to wonder about the fractional part of the whole. There is a reason to produce equivalent fractions. They use a variety of strategies. Mathematical questions arise.

Noticing how children are thinking about a problem, noticing whether they stay grounded in the context, tells the teacher whether or not the context is a good one. When the context is a good one, the children talk about the situation. When a problem is camouflaged school mathematics, children talk about numbers abstractly; they lose sight of the problem as they try to figure out what the teacher wants.

Carol's context had the potential for genuine mathematizing as her students cut up sandwiches and attempted to determine whether the portions were fair. The situation was meaningful to them. Finding equivalent fractions in the context was critical in order to be able to compare the portions. Big ideas, like *the whole matters* (that is, $\frac{1}{5}$ of $\frac{1}{2}$ is different than $\frac{1}{5}$ of the whole sub) and the *connection of fractions to multiplication and division* (that is, three subs shared with four children produces three fourths of a sub each, because $\frac{3}{4} = 3 \times \frac{1}{4}$), surface for discussion. As the class investigated fair-sharing scenarios, patterns appeared in their data, and these patterns triggered additional explorations. In contrast, the context in traditional word problems quickly becomes unimportant; children say "put the eighteen on the top" or "eighteen twenty-fourths" rather than "three fourths of the money." And once they have an answer to the "teacher's question," they see no reason to employ alternative strategies or to inquire further.

One could argue that if Susan had asked the children to find the reduced form in Josh's drawings or Annie's cubes, more learning would have resulted. For example, Josh might have circled three out of every four dollars, or Annie might have made her cubes into four groups and shown how three of the groups was equal to the eighteen. And a discussion around the connection between these solutions might have become rich. Probably this is true. One of Susan's problems is that she too readily went to an abstract algorithm for reducing. But does the context support the development of these alternative strategies? There is no reason to find the fractional part in the first place, and even less of a reason to reduce. The context is simply a contrived one to get the children to use the mathematics Susan wants. Susan's starting point is the discipline of mathematics, a body of knowledge she knows, and she is designing problems to get children to discover it. This is distinctly different from using rich contexts to support *the development of mathematizing.*

Finding Situations for Mathematizing

If the goal of mathematics instruction is to enable children to mathematize their reality, then situations with the potential to develop the ability to mathematize need to be carefully designed (or found). To encourage children to become mathematically literate—to see themselves as mathematicians—we need to involve them in making meaning in their world mathematically.

Situations that are likely to be mathematized by learners have at least three components:

1. The potential to model the situation is built in (Freudenthal 1973). Fair-sharing scenarios, working with measurements, increasing or decreasing portions in recipes, grocery and retail store scenarios, sharing money, calculating costs and savings, following stock losses and gains, collecting data and finding ways to organize them, all have the potential to develop mathematical modeling. Using the same model over time in different situations, and reflecting on the connection, supports the development of generalization.
2. The situation allows learners to *realize what they are doing.* It can be fictitious, but children are able to experience or imagine it and are able to think and act within its parameters (Fosnot and Dolk 2001). Children, as they share submarine sandwiches in equal portions, can picture or imagine the mathematics concretely and can check the reasonableness of answers and actions. (Putting eighteen over twenty-four to make $^{18}/_{24}$ of the money makes no sense, since in this context one is not concerned with the fraction spent but with the dollar amount left.) The Dutch use the term *zich realiseren*, meaning "to realize in the sense of to picture or imagine something concretely" (van den Heuvel-Panhuizen 1996).
3. The situation prompts learners to ask questions, notice patterns, wonder, ask why and what if. Inquiry is at the heart of what it means to mathematize. Questions come from interacting with the world around us, from exploring relationships, from trying to solve problems. When the problem is "owned," it begins to come alive.

Building in Constraints

Learners' initial informal strategies are not the endpoint of instruction; they are the beginning. Teachers must transform these initial attempts into more formal and coherent mathematical strategies and models. Although peer discussions and teacher questioning can lead students to restructure their initial ideas, building constraints into the context is often a more powerful means to that end.

Both Carol and Susan choose to focus on fractions and reducing because they are important mathematical topics. But we can also build potentially realized suggestions and constraints into contexts to further support development. For example, Carol can follow her initial submarine sandwich scenario with asking her students to create a chart for the cafeteria that will ensure that everyone will always get, say, $\frac{3}{4}$ of a sub for a field trip lunch. How many subs for a group of twelve children? twenty children? How many for a group of ten children, a messier number? A context like this is likely to bring up the use of ratio tables and learner-generated rules for equivalence.

Open vs. Closed Situations

Real learning is constructive and developmental. As children attempt to make sense of a situation and its context, they interpret, organize, and model it based on the ideas or strategies they have already constructed. They schematize and structure it so that it makes sense. Piaget (1977) called this process *assimilation*, meaning "to make similar." The process of assimilation has often been misunderstood as *a taking in*. Rather, it is *an acting on*. We act on experiences when we attempt to understand them, using strategies for interpreting, inferring, and organizing. We build new ideas on old ones or reformulate old ideas into new ones.

Learners will assimilate contexts in many ways. In every classroom, developmental differences will affect perceptions and strategies. And any new ideas constructed will be directly linked in learners' minds to *their* past ideas, because they arise from reorganizing the initial ideas.

In Carol's class, the students employ any number of ideas, inquiries, and strategies. The goal is not the same for everyone every day, but there is equal opportunity for everyone to learn because the situations and their contexts are so open. The submarine sandwich scenario offers many entry points for children, from drawing the subs and cutting them to make equal parts, to determining what to call pieces of pieces ($\frac{1}{5}$ of $\frac{1}{2}$, for example), to working with landmark fractions for comparison, to exploring fractions as common fractions or as the sum of unit fractions. Carol varies her questions to stretch and support individual children's learning.

Closed situations have only one possible strategy. Everyone is supposed to solve the problem in the same way, and learners are either successful or unsuccessful—they either get it or they don't. Open situations, crafted sophisticatedly with a didactical use of context, allow for and support developmental differences, and thus can facilitate mathematical development for everyone.

Word Problems vs. Context Problems

Word problems on the surface appear to offer many possible strategies by which to arrive at a solution. But because they are often designed with little context, they are usually nothing more than superficial, camouflaged attempts to get children to do the procedures teachers want them to do—procedures that have little to do with genuine mathematizing. And they often cause students to answer them in ways that fail to take account of the reality of the situations described (Verschaffel et al. 2000). Context problems, on the other hand, are connected as closely as possible to children's lives, rather than to "school mathematics." They are designed to anticipate and develop children's mathematical modeling of the real world. Thus, they encourage learners to invent genuine diverse solutions. In addition, context problems have built-in constraints in an attempt to support and stretch initial mathematizing. In this sense, their purpose is to promote the *development* of mathematizing. But is even this enough?

Context-Based Investigations and Inquiries

If genuine mathematizing involves setting up relationships, searching for patterns, constructing models, and proposing conjectures and proving them, then context must be used in a way that simultaneously involves children in problem solving *and* problem posing. Carol could simply have asked her students to figure out how much of a sub each child would get if four children were given three subs to share. This is a real situation, one that children could mathematize in many ways because they might divide the subs differently. They could count; they could explore naming pieces of pieces. But would children have noticed the pattern that Jennifer and John noticed? Would Jackie and Ernie have come up with their elegant comparison strategy? Would Michael, Gabrielle, and Ashleigh have constructed the equivalence of the summed unit fractions to common fractions? Would the problem be messy enough to support inquiry?

To allow the students to notice patterns, the situation and its context had to be open enough that patterns in data would appear. Piaget (1977) argued that setting up correspondences by learners is the beginning of the development of an understanding of relationships. Constructing a connection, a pattern, or a correspondence between objects fosters reflection. Learners begin to wonder why; they want to explain and understand the connections they notice. By building a problem with four situations, rather than one, and by asking her students to compare them, Carol opens the situation to become a genuine investigation rather than a problem, and the children can begin to construct relationships from the patterns they notice. But still this is not enough.

Carol must also facilitate the students' questions. As they raise inquiries, Carol gets excited along with them and deliberately gets them to discuss their ideas. She supports their inquiries by giving them the time and the materials to pursue them. If she had not facilitated this aspect of mathematizing—the problem posing—but instead had relied on a series of context problems to be solved (even when carefully structured day by day), she would not have developed in the children the ability to mathematize *their* lived world. Some children would have been lost along the way as the class as a whole moved from activity to activity. Instead, by using context-based *investigations* and by facilitating *inquiry* in relation to them, Carol involves her children in genuine mathematizing, in being young mathematicians at work.

TURNING CLASSROOMS INTO MATHEMATICAL COMMUNITIES

Knowing the difference between word problems, context problems, investigations, and inquiries, and knowing how to keep them open, helps Carol support each child. Understanding how to mold contexts is an important didactical tool for stretching each child. But understanding the role of context is not enough. Carol also makes her classroom a community in which her students can investigate and share with one another. Developing a community that supports risk taking and mathematical discussions is another critical pedagogical component for fostering real investigations and inquiries—real mathematizing.

The Edge Between the Individual and the Community

Teaching has two important and very different phases. At home, at night, we prepare for the next day. We replay the day just past, remembering the successes, evaluating the inquiries, celebrating the insights some of the children had, recalling the stumbling blocks and the struggles—all from the perspective of mathematical development, with a sense of the landscape of learning. Although our reflections begin with individual children, as we plan we shift our attention to the community—the whole class. Our intent is to keep everyone in the community moving—to move the community as a whole across the landscape toward the horizon. No matter what path a child is on, no matter where on that path the child is, we want to move that child closer to the horizon. Fortunately, we do not need to plan separate lessons for each child—nor could we. Instead we can focus on the community, thinking of contexts and situations that will be likely to move the community as a whole closer to the horizon. To that end, our lessons must be open and rich enough that each community member can enter them and be challenged.

The next day, in class, our role changes dramatically. We become a member of the community. We listen to and interact with the children. We try to understand what each child is thinking. We decide whether to ask for clarification. We pose questions that will cause children to think. We are intrigued with individual inquiries and solutions. We think about how members of the community can help one another, how they can build their ideas upon others' ideas. The night before, we are curriculum designers—designing the environment for the community. In class, we are researchers and guides. We journey with the children.

Therein lies our duality: we are community members, yet we plan for the community. We facilitate conversation around mathematical ideas and strategies for the community to consider. But, as a member of the community, we help develop the norms of what it means to prove something, of what counts as a solution, or a conjecture. We walk the edge between the community and the individual.

Facilitating Dialogue

Turning a classroom of between twenty and thirty individuals into a community is not easy: it's a structure very different from the classrooms most of us attended. Traditionally, dialogue in a classroom bounced from teacher to student, back to the teacher, then to another student. The teacher was there to question and give feedback. She stood at the front of the classroom; the learners were spread out in front of her.

In a "community of discourse" (Fosnot 1989), participants speak with one another. They ask questions of one another and comment on one another's ideas. They defend their ideas to the community, not just to the teacher. Ideas are accepted in the community insofar as they are agreed upon as shared and not disproved. The community develops its own norms for what it means to prove one's argument, for what stands as a mathematical problem, for how data get collected, represented, and shared. As a member of the community (but walking the edge), the teacher facilitates, monitors, and at times provides counterexamples and/or highlights connections to ensure that this dialogue supports genuine mathematical learning.

Several strategies can be helpful. After a student shares an idea, we can ask, *How many of you understand this point well enough to rephrase it in your own words?* (Or, as Carol did, "Who understands and can put in their own words where this group got the answer one tenth?") The students' responses tell us not only how many of them appear to understand but also *how* they understand, how they are schematizing, structuring, and modeling. Discussion cannot happen if the community is not considering the speaker's thinking. Because construction, not transmission, lies at the heart of learning, everyone is responsible for thinking about and commenting on one another's ideas. After several children have paraphrased an idea and we are confident that most students are participating, we can ask follow-up questions: *Does anyone have a question? Who agrees? Who disagrees? Does anyone have a different idea or a different way of thinking about it?* Questions like these keep the dialogue bouncing from student to student, from community member to community member.

Structuring Math Workshop

Investigations

When classrooms are workshops—when learners are inquiring, investigating, and constructing—there is already a feeling of community. In workshops learners talk with one another, ask one another questions, collaborate, prove and communicate their thinking to one another. The heart of math workshop is this: investigations are ongoing, and teachers try to find situations and structure contexts that will enable children to mathematize their lives—that will move the community toward the horizon. Children have the opportunity to explore, to pursue inquiries, and to model and solve problems in their own creative ways. Searching for patterns, raising questions, and constructing one's own models, ideas, and strategies are the primary goals of a math workshop. The classroom becomes a community of learners engaged in activity, discourse, and reflection.

Math Congress

After investigating and writing up solutions and conjectures, the community convenes for a "math congress." This is more than just a whole-group share. The congress continues the work of helping children become mathematicians in a mathematics community. Mathematicians communicate their ideas, solutions, problems, proofs, and conjectures to one another. In fact, mathematical ideas are held as "truth" only insofar as the mathematical community accepts them as true.

In a math congress, learners—young mathematicians at work—defend their thinking. Out of the congress come ideas and strategies that form the emerging discipline of mathematics in the classroom. The sociocultural aspects of this emerging discipline are directly connected to the community. What holds up as a proof, as data, as a convincing argument? What counts as a beautiful idea or an efficient strategy? How will ideas be symbolized? What is mathematical language? What does it mean to talk about mathematics? What tools count as mathematical tools? What makes a good mathematical question? What serves as a conjecture? All of these questions get answered in the interactions of the community. The answers arise from the sociocultural norms and mores that develop (Cobb 1996; Yackel 2001).

Once again we as teachers are on the edge. We must walk the line between the structure and the development of mathematics, and between the individual and the community. As we facilitate discussions, as we decide which ideas to focus on, we develop the community's norms and mores with regard to mathematics, and we stretch and support individual learners. We move the community toward the horizon, *and* we enable individuals to travel their own path.

We can structure math congresses in many ways. If we want to focus on a big idea or illuminate mathematical modeling, we can bring out the connections between different solutions and strategies. If we want to support the progressive development of strategies, we can direct the discussion from less efficient to more efficient solutions. Our goal is always to develop mathematizing—to promote shifts in thinking, to help learners develop mental maps. We focus on the community's journey, yet we work toward each student's construction of meaning.

Minilessons

A description of math workshop would not be complete without a few words about minilessons. Often we may wish to highlight a computational strategy, share a problem-solving approach, or discuss a historical proof. A ten-minute minilesson at the start of math workshop is a great way to do so. (Chapter 7 provides examples of many minilessons teachers have presented to develop mental math computation strategies.) In a minilesson, we as teachers take a more explicit role in bringing ideas and strategies to the surface. But once again we walk the edge. We put forth ideas for the community to consider, but we must allow individuals to construct their own meaning.

SUMMING UP . . .

Learning and teaching are interrelated; one does not occur without the other. Genuine learning is not linear. It is messy, arrived at by many paths, and characterized by different-size steps and shifts in direction. Genuine teaching is directed toward landmarks and horizons. The first epigraph to this chapter is a statement by the great mathematician Karl Gauss: "It is not knowledge but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment." As we learn, we construct. We near the horizon only to have new landmarks appear. As W. S. Anglin reminds us, "Mathematics is not a careful march down a well-cleared highway, but a journey."

Because learning is not linear, teaching cannot be either. If we as teachers have a deep knowledge of the landscape of learning—the big ideas, the strategies, and the models that characterize the journey—we can build contexts that develop children's ability to mathematize. By opening up situations into investigations and facilitating inquiry, we can support children's journeys along many paths.

But we need to walk the line between supporting individuals and planning for the community. Development of the class as a community is critical. In a community, trust and respect are shared by everyone. Traditionally, respect was reserved for the teacher: the teacher spoke, learners listened, and the teacher always had the last word. For a community to function well, all members must respect one another. Everyone's ideas deserve attention, and each person must be trusted to be responsible for the task at hand. Everyone must be trusted to be able to learn. In the beginning of the year, teachers need to work hard establishing routines and structures for math workshop. The learners in their charge must be led to trust that their ideas count, that their peers and the teacher really care about their thinking, that they will be given the time to explore different strategies and pursue their inquiries, that their questions and insights matter.

But community cannot be divorced from content. Mathematicians talk about mathematical ideas, not feelings or rules of behavior. They respect one another for the mathematical ideas they bring to the discussion. Learners, no matter how young, know when they are really being listened to. They know when they are learning and when they are not. They know when what they are doing is interesting, when it matters, and when it is simply about pleasing the teacher. When intriguing contexts are being explored and mathematical big ideas are being grappled with, engagement is high. Children can be mathematicians when teachers give them a chance to mathematize *their* reality, and trust that they can.