LESSON 1

ROD STAIRCASE

OBJECTIVE

Students will experience how to collect data, record them, and use them to make a prediction on the results for the general case.

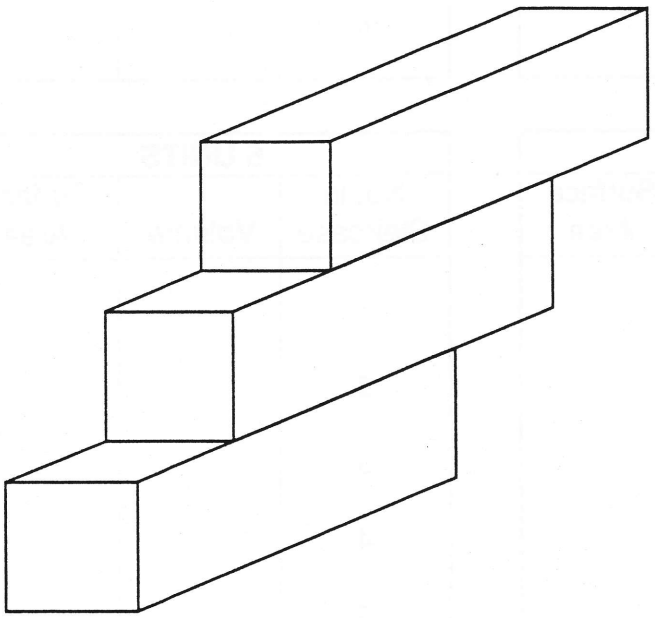
MATERIALS

Whether working individually or in groups, students will have sets of Cuisenaire rods with which to work.

EXPLORATION

The goal is for students to make staircases of a varying number of steps, each staircase the same color (each rod having the same length). Each step of the staircase is to be offset 1 unit. Be sure students understand that the area of the face of the smallest cube is 1 square unit, and its volume is 1 cubic unit. This cube is the standard by which all measurements are to be determined.

The teacher holds up a three-step staircase where each step is 3 units long. It should look like the following.



Distribute page 9, Build a Staircase, to each student or group.

BUILD A STAIRCASE

3 UNITS		
No. in Staircase	Volume	Surface Area
1		
2		
3		
4		
5		
6		
—		
n		

4 UNITS		
No. in Staircase	Volume	Surface Area
1		
2		
3		
4		
5		
6		
—		
n		

5 UNITS		
No. in Staircase	Volume	Surface Area
1		
2		
3		
4		
5		
6		
—		
n		

6 UNITS		
No. in Staircase	Volume	Surface Area
1		
2		
3		
4		
5		
6		
—		
n		

power of the creation of procedures and the need to judge the appropriateness or inappropriateness of a procedure in particular situations?

- Are the tasks posed engaging to students?
- Do the tasks posed require any reflection on, or interpretation, of results?
- Do the tasks posed keep students engaged with important concepts, skills, and procedures over time (distributed practice)?

The following sections show types of situations that have many of the desired characteristics implied by the questions raised.

Physical Investigations

Consider the following situation: a same-color staircase is made from Cuisenaire rods (see fig. 4.1.) Each time a rod is added to the staircase, it is offset by the space of a white (unit) rod. Suppose you are working with light green rods (3 units in length). Predict the volume and surface area (S.A.) that your staircase will have when you add the tenth rod,..., the twenty-fifth rod,..., the nth rod. Is the pattern the same if you build a staircase from rods of a different color (length)?

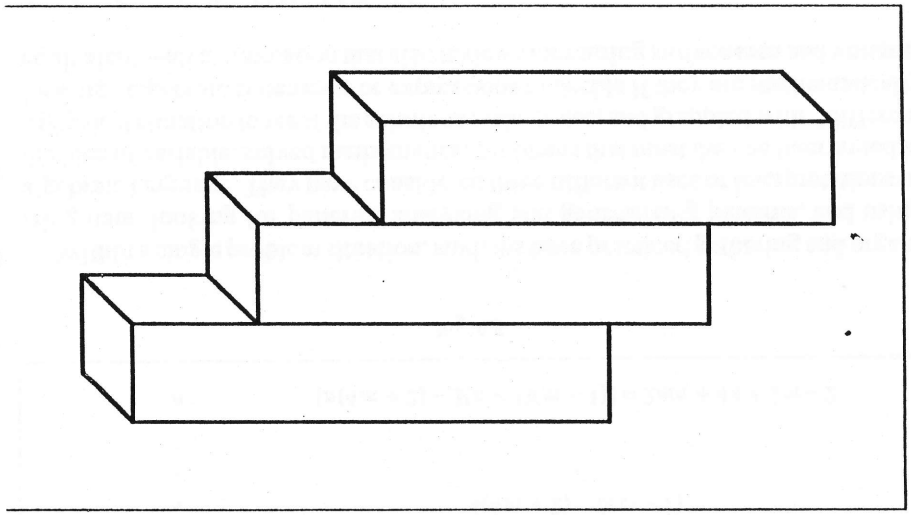


Fig. 4.1. Cuisenaire-rod staircase

Examples of data collected and organized for several different rods (fig. 4.2) show the richness of the situation. The pattern for volume is simply a multiple of the volume of the basic rod. The pattern for surface area is always an arithmetic sequence, but the rule is different for each basic rod. In each instance, variables are being used as a means of describing and generalizing a pattern in numbers. When we ask follow-up questions, we can probe other interpretations of the meaning of variable. For example, the question "What will the surface area be for a twenty-five-yellow-rod staircase?" asks the student to replace the variable in the expression with a member of the replacement set and to compute the value of the rule or function for that value of the independent variable. Next we can also ask problems of the sort, "Could the surface area ever be 625 for a staircase of yellow rods?" We are viewing the variable as standing for a specific unknown, namely, the solution to the equation $14n + 8 = 625$. If this solution gives a whole-number value of n, then the original physical problem posed has a solution, but if the solution to the mathematical equation is a fraction, the real problem has no solution.

Light Green Rods (3 Units in Length)

No. in Staircase	Volume	S.A.
1	3	14
2	6	24
3	9	34
4	12	44
.	.	.
.	.	.
.	.	.
n	$3n$	$10n + 4$

Purple Rods (4 Units in Length)

No. in Staircase	Volume	S.A.
1	4	18
2	8	30
3	12	42
4	16	54
.	.	.
.	.	.
.	.	.
n	$4n$	$12n + 6$

Yellow Rods (5 Units in Length)

No. in Staircase	Volume	S.A.
1	5	22
2	10	36
3	15	50
4	20	64
.	.	.
.	.	.
.	.	.
n	$5n$	$14n + 8$

Dark Green Rods (6 Units in Length)

No. in Staircase	Volume	S.A.
1	6	26
2	12	42
3	18	58
4	24	74
.	.	.
.	.	.
.	.	.
n	$6n$	$16n + 10$

Fig. 4.2. Data for staircase made from different rods.

Finally, the big question that raises the level of generalization is "Can we find a rule to predict the volume and surface area for a staircase n rods high built from any rod you choose?" In other words, does a super rule exist that lets you enter both the rod chosen and the number of rods in the stack to find the surface area or volume? The physical nature of this problem stimulates many ways of thinking and reasoning about the patterns observed. The situation provides a context against which mathematical conjectures can be tested for the sense that they make. Figure 4.3 presents two different ways of arriving at a super rule. The results lead to the problem of showing that the two different strings of symbols (the two rules) are mathematically equivalent.

Length	Surface Area for n Rods in Staircase
2	$2(4n + 1)$
3	$2(5n + 2)$
4	$2(6n + 3)$
.	.
.	.
.	.
m	$2[(m + 2)n + (m - 1)] = 2mn + 4n + 2m - 2$

No. in Staircase	Surface Area For Staircase Constructed With Rods of Length (Rods × Surface Area – Overlap)
2	$2(4m + 2) - 2(m - 1)$
3	$3(4m + 2) - 4(m - 1)$
4	$4(4m + 2) - 6(m - 1)$
.	.
.	.
.	.
n	$[n(4m + 2) - 2(n - 1)(m - 1)] = 2mn + 4n + 2m - 2$

Fig. 4.3

Within a single problem situation, students have practiced gathering and organizing data, looking for patterns, analyzing and generalizing patterns, and using algebraic language. They have considered three different uses or interpretations of the idea of variable, solved mathematical problems that must then be interpreted in a physical situation to see if the solutions make sense, and grappled with "different looking" algebraic statements or expressions to decide if they are mathematically equivalent—all in a situation that also reviews measuring surface area and volume.

Applications to Real-World Problems

Problems that allow the practice of skills in real-world contexts have the added advantage of showing the value of mathematics in our culture.

The Bouck family is going on vacation. The Boucks are planning to fly to Denver, Colorado, and then rent a car for eight days to drive through the Rocky Mountains to Yellowstone National Park and back to Denver. Mr. Bouck asked his daughter, Emily, to help him figure out the best deal on rental cars. They can choose from three car-rental agencies, whose terms are shown below:

U-Can-Rent-It	Good Deal Car Rental	Tri-Harder Car Rental
Daily rates	Daily rates	Daily rates
Mid-size	Mid-size	Mid-size
$38 per day	$42 per day	$48 per day
75 miles free per day; $0.32/mile over 75	100 free miles per day; $0.30/mile over 100	Free unlimited mileage
Full tank gas	Full tank gas	$2 per day for 4-door car.
		$12 one-time charge for fuel
		1/2-tank gas

In addition, Mr. Bouck belongs to a travel club that gives him 10 percent off the total bill if he rents from "Good Deal" and 5 percent off if he rents from "U-Can-Rent-It" but nothing off from "Tri-Harder." From which company should Emily tell him to rent?

One very valuable aspect to this problem is that its solution depends on the particular description that each student gives for the Bouck family's trip. Do they drive straight to Yellowstone National Park and back? Do they explore along the way and around the Yellowstone Park area? How many miles per day do they average over the eight days? The students could also be asked to describe more generally how to figure the bill for each car-rental agency (write the function) and to describe under what circumstances, if any, each dealer has the best rates. Graphing the three functions gives a picture that allows quick visual comparisons.

Reverse the Directions

The following examples of problems that can be used to maintain skills while deepening understanding use reversing the directions as a context for exploration. Algebra is often taught by giving an example and showing a procedure for solution followed by practice of like problems. This process can be reversed in ways that often require more open-ended thought on the part of the students. The first illustration of reversing is to ask students to create examples of specific kinds.

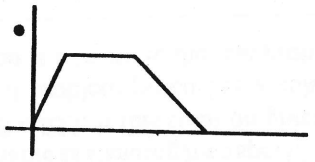
Give an example of—

- a quadratic equation with +2 as a root;
- a quadratic equation whose graph has no x-intercepts;
- three different lines that pass through the point (–2, 1).

Another kind of reversing is to give the mathematical model and ask students to write stories to fit the model.

Write two different stories that would fit the mathematical model given:

- $f(x) = 2x + 5$
- $y \leq x^2 + 3x - 5$



x	y
1	2.9
2	4.9
3	3.9

x^2