Chapter 3

Building Understanding of Algebraic Representation

The Border Problem, Part 2 (March 12)

Visualization . . . is one of the most rapidly growing areas of mathematics and scientific research. Learning to visualize mathematical patterns enlists the gift of sight as an invaluable ally in mathematical education.

STEEN (1990, 6)

Background of the Lesson—Cathy's Perspective

The Border Problem is the first problem in a sequence of lessons I designed for seventh graders. Students learn to use algebraic notation to represent growth patterns by relying on their ability to visualize what the pattern would look like at any stage of growth. As the unit progresses, students build proficiency with tables, graphs, symbolic rules, and verbal rules. While I knew that understanding the connections among these representations is crucial for attaining flexibility and competence with algebraic expressions (NCTM 2000), I had also learned from my research that the ability to move freely among these representations is essential for conceptual development of functions (Brenner et al. 1997; Hollar and Norwood 1999; Kieran 1992; Ritchhart 1997; Van Dyke and Craine 1997). And because functions are so important in higher mathematics, I also devised lessons that would develop explicit connections among the different representations to lay the groundwork for future understandings.

On the first day of the Border Problem, as seen in Chapter 2, the students generated different ways of visualizing the number of unit squares in the border of a 10-by-10 grid. They also began to think about how the six equivalent arithmetic expressions were alike and different and began to imagine stretching and shrinking the grid. I found the introductory lesson to be interesting and accessible for the students, but now the real work of connecting their thinking to the underlying

mathematics of generalization and multiple representations would begin. In this lesson, I planned to introduce students to four different representations (geometric, numeric, verbal, and algebraic) for the functional relationship between the length of the side of a grid and the number of unit squares in the grid's border.

In most curriculum materials involving growth patterns, the table of values, often called an *in-out table*, is normally the first representation that students are given; they fill in the table and then look for patterns. Yet I made a deliberate decision not to introduce this particular representation in the early lessons of this unit. Why did I make this decision? For many students, patterns of the dependent variable in an ordered table—what is added for each stage of growth—obscure the relationship *between* the variables. In other words, students often focus on the *pattern* rather than the *function*. While recursive analysis is certainly useful as a window into how a function behaves, I prefer to introduce this tool later in students' experiences. Early emphasis on the table of values can also make students think that finding the rule is the goal of the lesson. To the contrary, the aim of the lessons as I have envisioned them is for students to be able to write algebraic expressions that represent particular visualizations. The point is not to find the rule—especially a simplified rule!—but rather to represent symbolically what they *see*. The different visualizations provide a springboard from which students can grapple with equivalence and they bring life to what otherwise might be strings of meaningless symbols.

In order to express their methods algebraically, students would need to learn to use variables. I was fairly certain that any prior experience my students had had with variables was limited to evaluating variable expressions and/or finding the value of unknowns in simple equations. But NCTM (2000) states, "Students' understanding of variable should go far beyond simply recognizing that letters can be used to stand for unknown numbers in equations" (225). And from my research I had learned that the concept of variable is much more complex than I had realized, and that it "frequently turns out to be *the* concept that blocks students' success in algebra" (Leitzel 1989, 29). I discovered that misconceptions about variables are resistant to change and often are exacerbated by instruction and instructional materials.

The most pervasive misconception students have about variables, for example, is that letters are *initials* or *labels* (e.g., *a* as a label for apples rather than the *number* of apples) (Booth 1988; Pegg and Redden 1990; Wagner 1983; Wagner and Kieran 1989; Wagner and Parker 1993). I was somewhat dismayed to realize this, as in the past I had used *b* to represent the number of squares in the border. But the knowledge helped me be explicit that the independent variable in this problem would represent the *number* of squares in the border, no matter what variable I used. Students also commonly think that different letters in an expression

cannot represent the same number (Booth 1988) or that changing a letter changes the solution (Wagner 1983). With this last misconception in mind, I decided that I should not choose a variable for the students but that they all should select a letter of their choice.

The other helpful bit of research I found was that students' understanding of the equals sign as a "get the answer" symbol is an obstacle for seeing an expression as an answer (Wheeler 1996, 324). This misconception is understandable, since most (if not all) of students' experiences in arithmetic have taught them to interpret the equals sign as a signal to "write the answer" (Booth 1988; Kieran 1992; Kieran and Chalouh 1993; Wagner and Parker 1993) or that it means "makes" (Stacey and MacGregor 1997). So another of my challenges would be to help students understand that the expression itself was an answer.

Prior to the Video Selection

I chose to use Joe's method, 10 + 10 + 8 + 8, as a model for the four representations the students would be introduced to that day. I began the lesson by asking students to visualize a 6-by-6 square grid and think about what Joe would do to find the number of unit squares in its border. I made it clear that I was not as interested in *how many* unit squares were in the border as I was in *what Joe would have done* to figure it out. When Sarah responded, I recorded on the overhead what she said without including the total number of squares:

6-by-6: 6 + 6 + 4 + 4

Next I had the students imagine stretching the grid to 15 by 15. Could they use Joe's method for this size grid? They agreed that they could; Shelley volunteered that Joe would have added 15 + 15 + 13 + 13. I recorded this directly under what I had written for the 6-by-6 grid. Because these examples seemed so straightforward for the students, I decided to test their flexibility by using a much larger grid. I asked if they thought they would be able to use Joe's method for a 233-by-233 grid. Several students chorused 233 + 233 + 231 + 231, and when I asked them if they thought they could apply Joe's method to *any* size grid, they eagerly agreed that they could. This kind of experience gives students a glimpse into the power of functions and I find it very exciting to be with them when this happens.

I wrote 233 + 233 + 231 + 231 on the overhead and told the students that this was one way to represent and analyze the relationship between the length of the side of the grid and the number of unit squares in the border; I called it the arithmetic representation.

Arithmetic:

10-by-10:	10 + 10 + 8 + 8
6-by-6:	6 + 6 + 4 + 4
15-by-15:	15 + 15 + 13 + 13
233-by-233:	233 + 233 + 231 + 231

Figure 3–1.

Figure 3–1 shows what was written on the overhead projector at that point. I asked students to copy this into their notebooks; this would serve as a model for work they would do individually.

Next, I drew a "generic" square (see Figure 3–2) to represent Joe's method geometrically, coloring it as shown, while they copied it.

In order to make an explicit connection between the two representations they had learned so far, I went back to the arithmetic of the 15-by-15 grid (15 + 15 + 13 + 13) and asked the students where the 15 was in the generic square, and where the 13 was, and why. I continued with questioning like this in order to help establish students' flexibility with moving between the two representations. They seemed confident, and using Joe's method seemed clear, so I thought we were ready to tackle the verbal representation.

The verbal representation is the hardest for me to introduce, and it is tempting to go directly to the algebraic representation, which somehow seems easier! Once again, however, research guided my decision making. I had learned that verbal statements about functional relationships are critical for understanding (Brenner et al. 1997; Dossey 1997; NCTM 2000; Van Dyke and Craine 1997) and that it is important to make sure students verbalize generalizations of patterns before asking them to formalize those generalizations symbolically. In asking students to write a verbal rule, however, I wanted more than a direct translation; I wanted the students to remain cognizant of the geometric roots of each part of

Geometic:

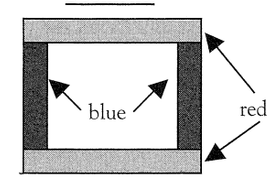

Figure 3–2.

their verbal rule. So, for example, instead of them just saying, "Subtract two," I wanted students to write something like, "Subtract two *for the overlapping unit squares on the corners*," to keep their thinking grounded in geometry. I hoped this would have the effect of building sense making in algebraic representation; I wanted them to see that each operation and each number means something particular and important in an expression.

I plunged ahead, asking students to think about how we could "write directions" that someone could easily follow to apply Joe's method. "What is the first thing Joe would do?" I asked.

Stephanie responded, "Take the number of unit squares on one side."

I was alert to the potential problems with the word *side*, as it could refer to either a position on the square (i.e., side as opposed to top or bottom) or *any* side. After clarifying that she meant on any side ("because all of the sides are the same length in a square"), I wrote this on the overhead transparency. "What next?" I inquired.

Sarah said, "Add that number to itself."

I asked where that made sense with the picture, and she responded, "For the top and bottom," which I recorded in parentheses. I asked students to think about how this sentence related to the generic grid I had drawn; Dana thought it was the "red part," and many students nodded in agreement.

"What next?" I asked. "What would Joe do next?" I asked the students to talk to each other in their small groups. After a few minutes I called on Colin, who said we should "subtract two from the other sides." I clarified the language a little, talking about how important it was to be clear when writing about mathematics, and recorded it this way: "Subtract two from each of the remaining sides. . . ." I then asked, "Why do we subtract two?"

Several students said, "For the corners." We had a lot of trouble here, getting the language just right. Some students wanted to say, "for the overlapping sides," but I pointed out that that didn't tell how much of each side was overlapping. With many students contributing (and some tuning out), we finally settled on this:

> Take the number of unit squares on one side. Add that number to itself (for the top and bottom). Subtract two from each of the remaining sides (for the overlapping ~~sides~~ unit squares on the corners). ~~Add the last two numbers.~~ Add these two numbers to the previous ~~number~~ sum.

The students then copied this into their notebooks under the heading, "Verbal." As the clip begins, I move to the algebraic representation by having the students consider how to "shorten" the verbal representation.

Watch "Building Understanding of Algebraic Representation: The Border Problem, Part 2," CD 1

Lesson Analysis and Reflection

When I asked how mathematicians might "shorten" the verbal description and still communicate Joe's method, Krysta seemed excited about the idea of using algebra. Her statement that mathematicians would "make the fifteens *x*s and the thirteens *y*s" to "put it all in letters" alerted me to a mistaken assumption that I had seen among my students in the past: that an algebraic expression replaces *all* of the numbers with letters. For example, in translating Sharmeen's method for the number of squares in the border ($4 \times 10 - 4$), a student might write $x \times y - x$ and then say that $x = 4$. These students, in mechanically replacing numbers with letters, show that they do not understand—yet—the language of algebra.

I immediately used Krysta's idea to introduce the term *variable*. Not all literal symbols take the role of variables, but literal symbols are so commonly called variables (whether their role is of a variable or not) that I felt compelled to say, as I was writing *variable* on the board, that the word "does not say everything about what they do." As I wrote the word, I heard Pam say, "I knew that!" I could tell the students liked the idea of doing algebra, which has special status as a rite of passage in middle school.

I then became uncertain about how to proceed. I had already decided that students should individually choose a variable in order to counteract the potential misconception, discussed earlier, that changing the letter changes the value of the expression. If everyone used the same variable, however, the class model for the algebraic representation would be more straightforward. I was momentarily unsure but decided on the former, figuring that the students would be able to easily adapt their own variable to what was written on the overhead. And, as expected, I got into a little trouble when I tried to do the class model; at first I was going to leave a space for the variable but then realized the obvious fact that I could not write an expression without one. So I chose *x*, while reminding the students that they should choose their own. The uncertainty that is evident in this vignette occurs every so often in my practice. The more I learn—about mathematics, about my students, about teaching itself—the more choices I have in the moment-to-moment decisions that a teacher must make in every lesson. The notion that there is one right, best way, what Chazan (2000) calls "a vocabulary of certainty," to approach a topic every time is foreign to me. Chazan continues, "Just as an overreliance on the categories of 'right' and 'wrong' inhibits discussion in mathematics classrooms, a technical vocabulary of certainty inhibits discussion of teaching practice" (153). I often

will purposely make different decisions in the same lesson with different classes. For me, it is so important to study and discuss with my colleagues the effects of different teaching moves; it is one of the things that makes teaching fascinating.

I also needed to decide whether to define the variable for the students or let them figure it out. I might have asked, "What does Joe need to know in order to find the number of unit squares in the border?" or "What should the variable represent?" and, indeed, I have followed that path with other classes. In *this* class and at *this* time, however, I decided to postpone the issue of how to define an independent variable in order to allow as much of the twelve or so remaining minutes for students to work together to use algebra as a "convenient shorthand for expressing ideas with which they have already grappled" (Schoenfeld and Arcavi 1988, 425–26).

Why didn't I just *show* students the correct way to write the algebraic expression? I could have written $x + x + (x - 2) + (x - 2)$, explaining clearly why the subtraction made sense. On its face, this approach seems more efficient; I could have believed that I had taught it and the students could have believed that they had learned it. But one of my major goals for these lessons was to have students learn to represent relationships algebraically—for themselves. I doubted that writing the expression for them would generate the tools and understandings they would need to solve problems that they had not yet seen. I have come to believe that relationships cannot be taught, but rather that "relationships are constructed, or made, by each individual" (Kamii and Warrington 1999, 83). Writing the algebraic expression for Joe's method requires students to consider how to represent two numbers that are related by a difference of two. Learning to represent that relationship is crucial for building sense making in algebraic representation.

I set the stage for them to write the algebraic expression by telling students that the first sentence (in their verbal representation) was important because it told "what Joe need[ed] to know to figure out the number of unit squares [in the border]." I proceeded to write, letting x be "the number of unit squares on one side," and I directed students to work together to "translate [the verbal representation] into an algebra expression," pointing out the verbal, geometric, and arithmetic representations that were visible as models on the overhead. What happened next was a direct result of my recent reading of Mark Driscoll's book *Fostering Algebraic Thinking* (1999). In his book, Driscoll recommends that teachers focus students' attention on what is staying the same and what is changing in order to help them learn how to build rules to represent functions. I directed the students' attention toward our arithmetic examples and asked them, "What's staying the same in this arithmetic?"

Pam volunteered, "You're always adding."

There was very little interest in this question, so I asked what was changing. Sarah noticed that "the first two numbers are the same and the last two numbers are the same." She did not mention that there was always a difference of two

between the first two addends and the last two addends, but again I made a decision not to press the students about this but rather to let them ponder it on their own. The impact of these questions surfaced later in the lesson.

As the students worked in their groups to write algebraic expressions, I visited a few groups to get a sense of how students were doing. My goal was not to correct students' errors or to put them on the right track but rather to ask questions and find out how they were thinking. This process helps me ferret out issues that would be good for class discussions or further work and it keeps students responsible for their thinking. As I walked around the classroom, I was drawn to the discussion Joe, Kayla, Pam, and Mindy were having; they were wondering whether they should use a different letter to represent the length of the remaining sides. An important convention in algebraic representation is that if there is a predictable relationship between numbers, we generally write one in terms of the other. This is a new idea for middle school students and a big leap for many because they rarely have the opportunity to encounter this idea in their textbooks.

The luxury of a videotaped lesson is that the camera offers a kind of eye in the back of the classroom. While the students were working in small groups, Antony, Sharmeen, and Kim had an interesting interaction that I learned about only while watching the tape. Sharmeen had a correct expression ($s + s + s - 2 + s - 2$) but erased it immediately when Kim said, "No. . . . " Watching the tape, I was dismayed at how quickly Sharmeen erased her work, that she did not ask why Kim thought her expression was wrong, and that Kim did not ask Sharmeen why she thought her expression made sense. It made me realize how easy it is to succumb to the assumption that group work is productive if there is a reasonably businesslike sound in the room. This incident underscored for me the importance of explicitly establishing and reinforcing class norms for group work; in this case, my students needed to establish the expectation that they would always give reasons for their answers.

This small interaction also demonstrated how students view what is simple versus what is complicated in algebraic expressions. Sharmeen read her expression ($s + s + s - 2 + s - 2$) to her group apologetically, saying she thought it was "complicated," not realizing that it was correct. Kim later said she also thought Sharmeen's expression was "really complicated." If I view this through their eyes, $s + s + k + k$ *looks* a lot simpler, maybe because there are less characters or because the 2s aren't there. I had, after all, asked students to "shorten" their verbal statements. Writing one number, $s - 2$, in terms of another, s, is indeed a different way of thinking about what is simpler.

In class, I overheard the end of this conversation and decided to stop the class. Hoping to promote a discussion that would raise this issue for everyone to consider, I began by asking Pam if she would be willing to tell the class her theory (that they needed a second variable). After Pam spoke, though, I realized that not everyone understood what Pam had meant. I called on Travis, who thought he

understood it but couldn't get the words out. So I went back to Pam to restate her idea then her groupmate Joe called on Stephanie to respond.

When Stephanie explained that it could be done with one variable, the timer rang, indicating five minutes to go in class; there was also the suggestion of widespread agreement with what she had said. I asked, "Why does that work? Why does that make sense?" Melissa's explanation focused on what was important—that all of the sides of the square grid were the same length, so we didn't need another letter. I wanted everyone to think about this: "When do you need another letter, and when don't you?" I knew that some of the tasks students would be tackling would require two independent variables, and I wanted to raise the issue and emphasize its importance. I was really surprised, then, when Pam commented that my earlier question about what was staying the same and what was changing had confused her. And she noticed something important: that "in the algebraic formula the things that are the same and different are not the same things that are the same and different on the numbers."

I was thankful not only that Pam had been able to put those thoughts into words but that she had so willingly expressed them. So often I am reminded of a perceptive statement by Daniel Chazan: "Student centered teaching makes the teacher all the more dependent on students. Not only were we dependent on our students to learn, but we also depended on them to help produce classroom interaction, or our lessons could come to a grinding halt" (2000, 120). It had not occurred to me that my question would lead some students into thinking that if you needed a different number, then you would need a different letter. But I was glad the light had shone so brightly that day on an issue that would come up again and again.

Sarah's algebraic expression at the end of the clip raised the issue of equivalence, which I took up with the students in another lesson. This lesson ended with Travis stating a correct algebraic expression to represent the relationship between the number of unit squares on one side and the number of unit squares in the border.

The Next Day

The next day, students worked in pairs to investigate and represent a second method arithmetically, geometrically, verbally, and algebraically. At the end of class, I collected their papers and read them carefully so that I could use their work as examples to help all of the students build their capacity for expressing relationships algebraically.

Postscript

About a month later I received an email from Kay, one of the students in this class. She was working on one of the last problems in our unit, which had different levels of difficulty. The level she was working on required three independent variables, one for length, one for width, and one for height. She wrote, "I have a question about the toothpick problem . . . If we were doing the expert level, we can only use 2 variables for each equation, right? Thanks."

This showed me how difficult the concept of variable really is for students. The idea that the length, width, and height of any rectangular prism are not related was not the issue for her; it was an issue of how many variables they were *supposed* to use. I was surprised by this question, but it helped me understand the complexity of the idea of dependence. Later, in her portfolio entry on this task, Kay wrote, "What this shows about me mathematically: This shows that I still wanted to use an extra variable, but I tried really hard not to."

Case Commentary—Jo's Analysis

Forming Algebraic Expressions

On the second day of the Border Problem, students made their way into the complex world of algebraic representation. We saw some initial conceptions and thoughts, most of them naïve and many of them expressed with uncertainty as students worked at the edge of their understanding. We saw a teacher making every effort to guide her students through some intricate terrain, making sure that she honored their thinking while also helping them avoid some pitfalls and misconceptions (Bouvier 1987; Nesher 1987). As often happened in this class, a discussion ensued in which students were clearly both curious and involved. The discussion addressed an important and often neglected area—that of *forming* algebraic expressions (Kieran 1992). Instead of starting algebra in the more conventional way—by telling students that, h, for example, represented hours and asking students to evaluate expressions when h was a number (Brown et al. 2000)—Cathy asked the students how they could shorten a written expression to communicate the same ideas "without all the writing." The students offered tentative ideas, giving us the opportunity to hear their first conceptions of algebra. A foundation was laid for the representational system that is at the heart of mathematics and that would aid students in their future inquiries into mathematical connections and relationships.

The Importance of Misconceptions

In the discussion that took place, students communicated a number of naïve conceptions. They said, for example, that it was OK to replace numbers with letters, but they didn't know what to do with an expression such as "subtract two." The students told us that $s + s + k + k$ was simpler than $s + s + (s - 2) + (s - 2)$. These ideas are reasonable; indeed, they make perfect sense given the experiences

these students had had to date. In some respects, $s + s + k + k$ is simpler and the students had learned that simplicity was good. They had yet to learn another mathematical appreciation—that capturing the relationships between numbers while avoiding the use of additional variables is mathematically important. Some people worry about students hearing misconceptions, thinking that students will remember the wrong idea rather than the correct one. But research tells us that students learn a lot when they consider competing ideas, even when some of them are wrong (Bransford, Brown, and Cocking 2000). When learners consider competing ideas, they engage in cognitive conflict and such conflict promotes learning more than the passive reception of ideas that are always correct and seem straightforward (Fredricks, Blumenfeld, and Paris 2004). We also know that it is important for teachers to consider what students know, paying "attention to the incomplete understandings, the false beliefs, and the naïve renditions of concepts that learners bring with them to a given subject" (Bransford, Brown, and Cocking 1999, 10). When Cathy encouraged students to share their initial ideas, she both gained access to the students' understandings, which she could then address through teaching, and she encouraged students to consider the validity and appropriateness of different algebraic expressions.

The discussion that took place in this lesson came from a relatively open question that Cathy asked: "How can we shorten Joe's written method?" The openness of the question gave students room to struggle with important ideas, such as ways to express a relationship and the need for new variables. Although the students struggled, they did not become anxious or disheartened. This was partly because Cathy had worked to establish classroom norms that valued exploration and even "wrong answers" (see student interviews on CD 2 for their reflections on this). The students were also confident and involved because their struggle was a collective one. We had the opportunity to watch the students work together as they navigated new terrain, learning new ideas, and their journey was, as always, an interesting one.

Focusing upon Functional Dependency

Cathy had a particular goal when planning her lesson: She wanted to encourage students to notice a relationship between the different numbers in each expression. In the early stages of the lesson the class was looking at three sets of numbers: $10 + 10 + 8 + 8$, $15 + 15 + 13 + 13$, and $6 + 6 + 4 + 4$. Cathy asked the following question (which she later regretted): "What's staying the same in this arithmetic?" She was looking for an appreciation of *functional dependency* (Kieran 1992)—an awareness that there was a relationship between the two different numbers in each expression. It is not surprising that Cathy was hoping for such awareness; she wrote in her background notes that she did not want students to

x	y^2
1	1
2	4
3	9
4	16

Figure 3–3.

begin this lesson by listing numbers in a table looking down the columns for recursive insights. For example, with the function $y = x^2$ students often form lists and consider each vertical column of numbers separately, saying that the first column (x) goes down in ones and the second (y) adds two more each time ($+ 3$, $+ 5$, $+ 7$). (See Figure 3–3.) Cathy wanted students to see more and to start to become accustomed to finding relations between members of the domain and their image. But the students did not communicate such an awareness; when Cathy asked what was the same in the expression, they offered answers such as "You are always adding." This was true, but it did not convey what Cathy was looking for. What should a teacher do in such a situation? One possibility would have been for Cathy to ask the same question again or to ask the question in a different way. When Sarah said, "Well, the first two numbers are the same numbers and the last two numbers are the same," Cathy could have asked Sarah about the relationship between them. But she chose to curtail her line of questioning and to move along, asking students to write an algebraic expression. Later Cathy tried to probe students about the relationship again, when Stephanie said that "you could just keep the same letter," Cathy asked, "Why does that work?" By the end of the lesson students were starting to form an awareness of the relationship articulated by Melissa, who told us that "all the sides are the same lengths, so that when you use one letter, you have to subtract two." But the lesson was drawing to a close at this point and the ideas were left to be developed on another day.

Was the question "What is staying the same?" the wrong one to ask at that time? Although Cathy did not get an answer she was looking for, I think it was a good question, partly because she asked the students to think in important ways about mathematical relations and they will have been encouraged to do such thinking, and also because Cathy was modeling for students the practice of asking mathematically worthwhile questions (Driscoll 1999) of their work (see Chapter 6).

The Role of Teacher Questions

In this extract we saw a teacher working to develop a critical concept through questioning. The concept was that of functional dependency—the relations between members of the domain and their image. Many research studies (e.g.,

Hiebert and Wearne 1993) have shown that teachers often fall into a trap of asking low-level questions that are not especially demanding and do not target key concepts. In a research study at Stanford, our team[1] investigated different high school mathematics approaches, monitoring students through four years of high school as they experienced different approaches (Boaler 2003a). One of the findings from that study was the critical role played by teachers' questions in the establishment of particular instructional environments as well as the mathematical *directions* of lessons (Boaler and Brodie 2004). As students work on mathematics problems in lessons, they traverse different sections of the mathematical terrain (Greeno 1991). Sometimes their paths are rather narrow—they may be led down a series of steps, without ever stopping to look at the broader landscape around them. Sometimes students wander unproductively around, never getting a sense of where they are in the terrain. The questions teachers ask guide students through particular pathways in the mathematical environment. In our analyses of lessons that start from the same task, we find that some teachers ask surface questions that do not take students deeper into mathematical issues; we think of those students as walking on a path that surrounds a beautiful forest without ever stepping into the forest to look at the trees. Other teachers ask questions that are more probing but that do not build carefully toward key concepts. We think of these students as stepping in and out of the forest, catching glimpses of trees and flowers but not learning where they are in relation to each other or how they may navigate their way through the forest. Other teachers ask questions that target key concepts and build carefully to enable students to find their way around. Those students experience the forest fully—they walk through, looking at the trees and flowers, and they also climb some trees and look at the whole terrain, getting a sense of where they are. The initial tasks that teachers use are critical in setting up the particular terrain that students will explore, but the questions that teachers use to guide students become the pathways that students walk along and that shape their experience of the terrain.

In our observations of hundreds of hours of lessons, we have noticed that teachers ask a range of question types. Previous analyses of question types have tended to divide questions rather simplistically into open and closed questions, or higher- and lower-order questions (for exceptions, see Hiebert and Wearne 1993; Driscoll 1999). But such categorizations do not seem to capture the nuance of the teaching act. Our categories are derived from an analysis of practice—we did not invent the categories *a priori*; we studied different examples of teaching and attempted to describe and name the different types of questions we recorded (see Table 1). Some of our category names draw on work by Driscoll (1999).

[1]Jo Boaler, Karin Brodie, Jennifer Dibrienza, Nick Fiori, Melissa Gresalfi, Emily Shahan, Megan Staples, and Toby White

Table 1: Teacher Questions

Question Type	Description	Examples
1. Gathering information, checking for a method, leading students through a method	Wants direct answer, usually wrong or right Rehearses known facts or procedures Enables students to state facts or procedures	What is the value of *x* in this equation? How would you plot that point?
2. Inserting terminology	Once ideas are under discussion, enables correct mathematical language to be used to talk about them	What is this called in mathematics? How would we write this correctly mathematically?
3. Exploring mathematical meanings and relationships	Points to underlying mathematical relationships and meanings. Makes links between mathematical ideas	Where is this *x* on the diagram? What does *probability* mean?
4. Probing; getting students to explain their thinking	Clarifies student thinking Enables students to elaborate their thinking for their own benefit and for the class	How did you get ten? Can you explain your idea?
5. Generating discussion	Enables other members of class to contribute and comment on ideas under discussion	Is there another opinion about this? What did you say, Justin?
6. Linking and applying	Points to relationships among mathematical ideas and mathematics and other areas of study or life	In what other situations could you apply this? Where else have we used this?
7. Extending thinking	Extends the situation under discussion, where similar ideas may be used	Would this work with other numbers?
8. Orienting and focusing	Helps students focus on key elements or aspects of the situation in order to enable problem solving	What is the problem asking you? What is important about this?
9. Establishing context	Talks about issues outside of math in order to enable links to be made with mathematics at later point	What is the lottery? How old do you have to be to play the lottery?

Some of these questions are worth explaining further. Type 1 questions (gathering information) are extremely common in mathematics classrooms. In our analyses of teachers using different approaches, these questions occurred frequently in classrooms using traditional and reform approaches, but they were particularly common in the more traditional classrooms. Indeed, analyses of more than twenty hours of lessons revealed that 95 percent of questions in the traditional classrooms were type 1. Teachers using reform approaches asked many more probing (type 4) questions; when students told them something, these teachers often asked how they knew or asked them to explain further. These are important questions, requiring that students justify and reason. But the question type that is arguably the most important of all—type 3, targeting key concepts—was observed very rarely. We called this question type "exploring mathematical meanings and relationships." Such questions orient students to the central mathematical ideas. They do not necessarily follow up on students' ideas; they often come from the teacher, and they serve a very particular and deliberate purpose: challenging students to consider a critical mathematical concept. Despite its importance, this type of question could not be used exclusively, and the range we have witnessed is very useful in helping students develop understandings, manipulate methods, learn vocabulary, and so on. But the questions that target mathematical meanings and relationships are critical, and surprisingly rare.

While the aim of this lesson was to express relationships algebraically and to begin to use algebraic variables, a central concept was that of functional dependency. A number of Cathy's questions were those we would code as type 3, as they targeted this concept; for example:

> What's staying the same in this arithmetic?
> Do you think you need to create another letter?
> When do you need another letter and when don't you?
> Why don't you need another letter in this case?

These questions are not about vocabulary, they are not asking for the execution of a procedure, and they are not assessing something *related to* the functional dependency, such as some arithmetic involved; rather, they directly target the concept. It is also noteworthy that Cathy had planned to ask students these questions as she prepared the lesson. Cathy knew that when she asked students to find shorter ways to write Joe's method, they would begin to use algebra and that they would stumble across the important issue of when to use more than one variable. While she may not have planned the exact wording of her questions, Cathy's careful planning, her depth of content knowledge, and her knowledge of student conceptions, ideas, and understandings combined to form these questions. Knowing to ask a question such as "when do you need another letter and when don't you?"

is an example of pedagogical content knowledge, which derives from a number of sources, including—in Cathy's case—the reading of texts on student understandings, reflection on student thinking, and the act of careful lesson planning. These questions played an important role in the lesson—they took students into critical mathematical territory and enabled them to consider the relationships there. They may not have been the best options and readers may want to consider what would have been a more ideal trajectory for the lesson, but these types of questions play a significant role in shaping the environment of the class and taking students into important mathematical terrain. I have highlighted the questions asked in this particular lesson, but teacher questions are important to all of the cases in this book and readers may wish to focus upon the questions asked in different cases.