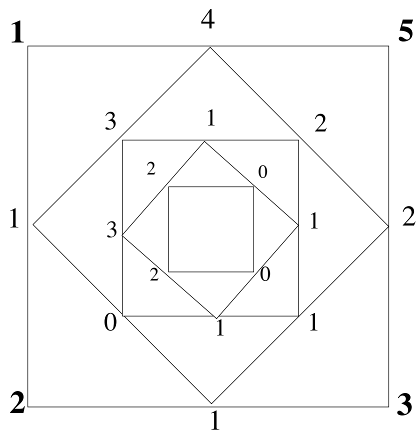
The Four-Number Game

by Daniel Shapiro, 2005

Choose four numbers and place them at the corners of a square. At the midpoint of each edge, write the difference of the two adjacent numbers, subtracting the smaller one from the larger. This produces a new list of four numbers, written on a smaller square.

What happens when this process is repeated?

Here are a few steps, starting with the four numbers 1, 5, 3, 2 around the largest square, and proceeding inwards. Once you see how it works, it's easier to display the game more compactly as a table:



1	5	3	2
4	2	1	1
2	1	0	3
1	1	3	1
0	2	2	0
2	0	2	0
2	2	2	2
0	0	0	0

After seven steps the numbers become all zeros. Let's try two more examples.

1	3	8	17
2	5	9	16
3	4	7	14
1	3	7	11
2	4	4	10
2	0	6	8
2	6	2	6
4	4	4	4
0	0	0	0

1	2	2	5
1	0	3	4
1	5	1	3
2	2	2	2
0	0	0	0
0	0	0	0

Each example ends with a row of zeros after a few steps. Take a couple of minutes to try out a few 4-number patterns for yourself. . .

Are there any examples that don't end with a row of zeros?

One way to investigate this question is to generate lots of examples. We could ask everyone we know to work out fifty examples. Or one of you could write a computer program to compute examples. But lists of examples can never prove that the process will *always* end in a row of zeros.

To find a proof we start by defining terms more carefully. If $Q = (a, b, c, d)$, the derived row Q' is obtained by taking differences, ignoring minus signs. The first entry of Q' will be either $a - b$ or $b - a$, whichever one is not negative. That entry is the absolute value $|a - b|$. With this terminology, if $Q = (a, b, c, d)$, the derived row is:

$$Q' = (|a - b|,\ |b - c|,\ |c - d|,\ |d - a|).$$

If we analyze the general situation directly, the cases and sub-cases proliferate: Which of the numbers is largest? Which of the differences is smallest? We use a more indirect approach.

OBSERVATION. If $Q \neq (0, 0, 0, 0)$, then Q' seems to be smaller than Q.

If this is always true, we can prove that our game must end in a row of zeros. For suppose $Q = (a, b, c, d)$ is given with non-negative integer entries. Repeat the process several times, obtaining rows Q', Q'', Q''', Q'''', . . . By the Observation the entries in those derived rows get smaller and smaller. Eventually they become zero, since a decreasing sequence of non-negative integers cannot go on forever.

But is that Observation true? In the examples the numbers get smaller as the game is played, but what exactly does it mean for one row to be "smaller" than another? For instance, is (1, 0, 4, 12) smaller than (3, 3, 5, 4)? Here's another example:
If $Q = (4, 0, 0, 0)$ then $Q' = (4, 0, 0, 4)$. Here the size did NOT decrease (whatever measure of size we use).

To clarify the "size" of a row, let's consider the maximal entry:

If $Q = (a, b, c, d)$, let **max(Q)** be the largest of the four numbers in Q.

Since our process uses subtraction, the largest number in Q cannot increase. In mathematical terms, this says:

If Q' is derived from Q then: $\max(Q') \leq \max(Q)$.

Those maximal values might be equal (as seen when $Q = (4, 0, 0, 0)$). That can happen only when there is a zero in the row Q. Since equality of maximal entries can happen we have to work a little more, running the game a few steps.

CLAIM: **For any row R, at least one of the rows**
R or R' or R'' or R''' or R'''' has all entries even.

This Claim is proved by considering a few cases. Write "e" for an even number and "o" for an odd number, to track different cases. For instance the row $Q = (4, 2, 1, 1)$ becomes (e, e, o, o) and we find the derived row must be: $Q' \approx (e, o, e, o)$. (Why?) Similarly, $Q'' \approx (o, o, o, o)$, and $Q''' \approx (e, e, e, e)$, which is all even. There are several more e & o cases to work out, but we leave them for you to investigate.

Now we can prove that the game must eventually stop at (0, 0, 0, 0) for any initial row Q, no matter how large the entries of Q are. To start, run the game for a few steps until reaching some derived row S which is all even. (Using the Claim.) That row can be written as $S = (2w, 2x, 2y, 2z)$ for some whole numbers w, x, y, z. Let $T = (w, x, y, z)$ which is just ½S. The steps of the game applied to S exactly match the steps applied to T. (Why?) Then to analyze the game we can replace S by T. Note that $\max(T)$ is definitely smaller than $\max(S)$, at least if S was non-zero. Then the previous idea works: Continue the game, but each time we reach an all even row, factor out another 2.

If a row of zeros never appears, the maximal entries of the factored rows provide a decreasing list of positive integers that never ends. That's impossible! QED

That proves that every 4-number game of whole numbers must eventually stop. But there are still many questions to investigate. Here are a few for you to think about:

1. How many steps are needed? Is there some row that needs 20 steps to get to zeros? Is there one that needs 100 steps?
What's the longest number of steps needed for a row whose entries are all < 100 ?

2. What about other row sizes? For instance, what happens with rows of three numbers? Or with rows of five or six? What sorts of patterns occur as those game continue?

3. What happens when more general numbers are used in the four-number game? For instance, work out the game for $(0, 1, 6, \pi)$. Surprisingly this one goes to zeros in only four steps. Investigate some other non-integer examples. What happens?
For those cases our proof that the game ends in zeros no longer works. (Dang!) Are there any cases when the four-number game is infinite?

Note: Dozens of technical papers have been written about the four-number game. Those patterns are also called "Ducci sequences" after the mathematician who invented this game in the 1930s.

Here are a few references:

M. Lotan, A problem in difference sets, *Amer. Math. Monthly* **56** (1949) 535-541.
(Proves the general result for 4 real numbers.)

E. Berlekamp, The design of slowly shrinking squares, Math. Comp. 29 (1975), 25-27.
(Same result with fewer details.)

L. Meyers, Ducci's four-number problem: a short bibliography, *Crux Math.* **8** (1982), 262-266.

W. Webb, The length of the four-number game, *Fibonacci Quart.* **20** (1982) 33-35.
(The row $(t_n, t_{n-1}, t_{n-2}, t_{n-3})$ has a game of length about 3n/2, and no row involving numbers of that size can last longer. Here t_n is a Tribonacci number!)

R. Brown and J. Merzel, The length of Ducci's four-number game, *Rocky Mtn. Math J.* **37** (2007) 45-65.

A. Behn, C. Kribs-Zaleta, and V. Ponomarenko, The Convergence of Difference Boxes, *Amer Math Monthly* **112** (2005) 426-439.