"Where do you get your ideas?" is a question that every writer encounters all too often, and writers of problems for Math Teachers' Circles are no exception. To paraphrase Robert Pirsig's advice about painting in *Zen and the Art of Motorcycle Maintenance*, "You want to know how to write the perfect problem? It's easy. Make yourself perfect and then just write naturally." That's perhaps the most honest—and most useless—possible advice on the topic. It does point in some productive directions, though. What you want to do is not so much learn how to create good problems as to become the kind of person who notices good problems all around you, and to immerse yourself in a culture where you live in a higher density of such problems.

How do you learn to notice those suitable problems when they fall into your lap? For one thing, you need to cultivate the habit of playing with mathematics when you come across it. What are different ways you could look at it? Could you explain some aspects of it to a five-year-old? Can you reach important insights by doing experiments a middle school student could understand? Can you represent the idea in a different way? For instance, the geometric and number-theoretic ideas behind the game of SET were around for a very long time, but the playfulness encouraged by the game and the novel way of representing those ideas led to a lot of great MTC problems[1] and, ultimately, completely novel mathematics.

While we're looking for problems, we need to keep in mind that we're not hunting down exercises. As Paul Zeitz tells us, "Exercises may be hard or easy, but they are never puzzling" – we are supposed to know already how to approach them. A problem presents us with a novel challenge, where we're not sure what tools we need. By this definition, solving a Sudoku is generally an exercise; it's only the times when we're stuck and need to find a new approach, or a new way to put together old approaches, that qualify as problems. Of course, a simple exercise for an expert might be quite a challenging problem for a beginning solver who doesn't have the same library of techniques and strategies!

That novel challenge means that an intriguing problem feels **perplexing**. Good mathematics should be a tool for resolving perplexity! It's part of human nature to want to solve a puzzle. On the other hand, a puzzle can easily be too easy, so it's just a time waster, or too hard, so that it feels frustrating. So, an important part of the session leader's job is to put together the problem in such a way that it's **tuned to the audience**. Even more importantly, the problem should have plenty of **easy entry points**, so that people at almost any level can get started, with some early examples that contribute to understanding of the depth that will come later. But the learning doesn't come without some amount of challenge and discomfort. "Stuckness shouldn't be avoided," as Pirsig

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CLASSROOM CONNECTION
CLEAR & AMBIGUOUS
EASY ENTRY POINTS
BLEM" PROBLEM
by Joshua Zucker

advises. "It's the psychic predecessor of all real understanding." On the way to that stuckness, though, the introduction needs to be **concise and accessible**, to get everyone involved.

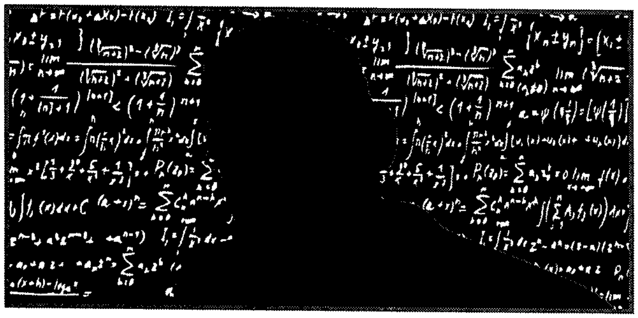
In addition, most in the MTC community agree that good problems should be **both clear and ambiguous**, which seems like about as blatant a contradiction as one could have. Perhaps what we mean is that the communication should be clear but the question might not be. We shouldn't be afraid to ask non-mathematical open-ended questions like "What happens?" or "What can you see here?" or to create a problem situation where participants then produce their own questions. The art of **problem posing** can be just as much a part of our sessions as the art and craft of problem solving. A good, deep problem leads to bigger problems and generates more questions of its own. So, in a further contradiction, our session might end with only a **partial conclusion**, where we feel satisfied about something having been resolved but perhaps with many further questions remaining to be answered.

The introduction of a perplexing problem can feel almost like a code you need to break[2] or the beginning of a good mystery novel. In a mystery novel, though, the clues are deliberately hidden and the detective may put them together in a way that seems almost magical. In mathematics, we want our participants to learn the secrets of our detective magic!

Thus, a good problem needs to **illuminate strategies** and techniques so that solvers come away with new tools that they can use to solve future problems. Moreover, they gain an appreciation for novel ways to combine those tools.

We can **connect** disparate problems by means of strategies, and conversely we can connect different areas of mathematics by means of a problem. Sometimes problems that are on the surface quite unrelated[3] turn out to have some deeper idea connecting them. As mentioned above, the game of SET connects geometry, number theory, combinatorics, linear algebra, and more. Many problems have both numerical and geometric interpretations, where problems on the grid[4] turn out to lead to deep number-theoretic ideas, or where a familiar multiplication table has a geometric interpretation[5] that leads to deeper understanding. **Multiple representations** are one way of generating connections and making new discoveries, as well as giving new insight to taken-for-granted fundamentals like place value[6,7].

These connections and multiple representations often lead to one of the best ways to make a session engaging: **surprise!** The best kinds of surprises are the sudden emergence of a pattern when work on a problem is organized a certain way or the sudden "obviousness" of a difficult fact when it's looked at from a new perspective. Untie ropes using arithmetic of fractions?[8] Do number theory and triangle geometry



by pouring water?[9] After some MTC experience, these kinds of surprises turn out to be, well, surprisingly common. So, cultivate a sensitivity to this kind of solution. Peter Winkler's books, as well as Martin Gardner's and Ian Stewart's, are excellent resources for problems with a great surprise.

To keep people engaged in working on the problems, it also helps to have a few key planned **landmarks** along the way. These can be "A-ha!" moments of breakthroughs, conclusions about one aspect of a problem, "Oops!" moments where something that everyone was assuming turns out not to be true, or "Now what if?" moments where variations on a theme can be explored. These landmarks can also be summaries of key discoveries. These key points help make it possible for participants to leave satisfied with their different levels of understanding.

Perhaps the question that most distinguishes the approach of various Circles in choosing problems is the extent to which our sessions should have a **classroom connection**. This can take many forms. For example, the topic can be directly related to middle school content standards, such as fractions[10], even if our exploration of it goes far beyond the standards. Or, we can choose a topic that embeds skills from the standards in a very non-standard way, like Rational Tangles[8]. There are advantages to this: teachers may be able to use a form of the material in their classroom as part of their normal lessons. But there are also disadvantages: teachers with their "teacher hat" on may end up feeling more like they're at work instead of playing with mathematics, and they may think more about how to communicate the results to their students instead of generating more results on their own. Whether or not there is a direct connection to content standards, good MTC problems should always have **mathematical practices** as a primary focus.

For many writers and solvers, one of the most appealing features of great problems is a compelling **story or history**. This makes the problems more memorable, and makes us all more a part of the culture of problem solving with its shared folklore. It also makes it more likely that participants will "leave the session **humming the problems**," as Ravi Vakil describes the goal of an author. There's no more satisfying compliment than to find that the participants have continued working on the problem long after the end of the session!

Links and Resources

1. The Game of SET, B. Conrey and B. Donaldson.
2. Codes, Ciphers and Secret Messages, M. Beck.
3. The Power of Wishful Thinking, A. Rodin, and Two is Better Than Infinity, P. Zeitz.
4. Grid Luck, T. Shubin.
5. Multicolored Multiplication Table, D. Millar.
6. Coins in Twoland, J. Zucker.
7. Exploding Dots, J. Tanton.
8. Conway's Rational Tangles, T. Davis.
9. After the Jug Band, E. Johnston, and The Three Jugs Problem, J. Zucker.
10. Fractions, Decimals, Ratios, Rates, Percents, Proportions, T. Shubin, and Fractions and Decimals, T. Davis.

For links to these resources and more, visit us online at http://mathteacherscircle.org/resources/sessionmaterials.html.