



You were asked to explain 10 hypothesis tests above. For each of the tests (from the one sample z-test to the Shapiro-wilks test), go into further detail about the following aspects of the tests:

1. What is the distribution of the test statistic?
2. How are the critical values calculated from the distribution?
3. How are the p-values calculated?

Below is a detailed breakdown of the **distribution of the test statistic**, **calculation of critical values**, and **calculation of p-values** for each of the ten hypothesis tests:

1. One-Sample Z-Test

1. Distribution of the Test Statistic

- The test statistic follows the **standard normal distribution** (Z-distribution) under the null hypothesis:

$$Z = \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}}$$

where σ is the known population standard deviation.

2. How Critical Values Are Calculated

- The critical values are found using the **standard normal distribution table (Z-table)**.
- For a significance level α , the critical values correspond to:
 - **Two-tailed test:** $Z_{\alpha/2}$ and $-Z_{\alpha/2}$.
 - **One-tailed test:** Z_{α} or $-Z_{\alpha}$.

3. How P-Values Are Calculated

- The p-value is computed as:
 - **Two-tailed:** $p = 2P(Z > |Z_{\text{obs}}|)$.
 - **One-tailed:** $p = P(Z > Z_{\text{obs}})$ or $p = P(Z < Z_{\text{obs}})$.
 - The p-value is obtained from the standard normal table.
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2. Two-Sample Z-Test

1. Distribution of the Test Statistic

- The test statistic follows the **standard normal distribution**:

$$Z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

2. How Critical Values Are Calculated

- The critical values are obtained from the **Z-table**, using the same procedure as the one-sample z-test.

3. How P-Values Are Calculated

- The p-value is computed similarly to the one-sample z-test, using the standard normal table.
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3. One-Sample T-Test

1. Distribution of the Test Statistic

- The test statistic follows a **t-distribution with (n - 1) degrees of freedom (df)**:

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

2. How Critical Values Are Calculated

- The critical values are obtained from the **t-distribution table**, based on the desired significance level α and $df = n - 1$.

3. How P-Values Are Calculated

- The p-value is calculated from the **t-distribution** using statistical tables or software.
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4. Two-Sample T-Test

1. Distribution of the Test Statistic

- If equal variances are assumed (**pooled t-test**):

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{s_p^2 \left(\frac{1}{n_1} + \frac{1}{n_2} \right)}}$$

- Degrees of freedom: $df = n_1 + n_2 - 2$.

- If unequal variances are assumed (**Welch's t-test**):

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

- Degrees of freedom: Approximate using **Welch's formula**.

2. How Critical Values Are Calculated

- Use the **t-distribution table** with df for the given significance level.

3. How P-Values Are Calculated

- The p-value is obtained from the **t-distribution table** or statistical software.
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5. Paired T-Test

1. Distribution of the Test Statistic

- The test statistic follows a **t-distribution** with $(n - 1)$ degrees of freedom:

$$t = \frac{\bar{d} - \mu_d}{s_d / \sqrt{n}}$$

2. How Critical Values Are Calculated

- Use the **t-table** with $df = n - 1$.

3. How P-Values Are Calculated

- Computed from the **t-distribution table** or software.
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6. F-Test for Variance Comparison

1. Distribution of the Test Statistic

- The test statistic follows an **F-distribution**:

$$F = \frac{s_1^2}{s_2^2}$$

- Degrees of freedom: $df_1 = n_1 - 1$, $df_2 = n_2 - 1$.

2. How Critical Values Are Calculated

- From the **F-distribution table**, using df_1 and df_2 .

3. How P-Values Are Calculated

- Computed using the **F-distribution table** or software.
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7. ANOVA (F-Test)

1. Distribution of the Test Statistic

- The test statistic follows an **F-distribution**:

$$F = \frac{MSB}{MSW}$$

- Degrees of freedom: $df_1 = k - 1$, $df_2 = n - k$.

2. How Critical Values Are Calculated

- From the **F-table** for the given df_1 and df_2 .

3. How P-Values Are Calculated

- Computed using the **F-distribution**.

8. Chi-Squared Test for Independence

1. Distribution of the Test Statistic

- The test statistic follows a **chi-squared distribution**:

$$\chi^2 = \sum \frac{(O_{ij} - E_{ij})^2}{E_{ij}}$$

- Degrees of freedom: $df = (r - 1) \times (c - 1)$.

2. How Critical Values Are Calculated

- From the **chi-squared distribution table**.

3. How P-Values Are Calculated

- Computed from the **chi-squared distribution**.

9. Shapiro-Wilk Test for Normality

1. Distribution of the Test Statistic

- The test statistic (W) follows a **Shapiro-Wilk distribution**, with values close to 1 indicating normality.

2. How Critical Values Are Calculated

- No direct critical value calculation; use tables or software.

3. How P-Values Are Calculated

- Computed using an **empirical distribution** based on sample size.

Summary Table

Test	Distribution	Critical Values	P-Value Calculation
One-Sample Z	Normal	Z-table	Z-table
Two-Sample Z	Normal	Z-table	Z-table

Test	Distribution	Critical Values	P-Value Calculation
One-Sample T	T-distribution ($df = n - 1$)	T-table	T-table
Two-Sample T	T-distribution ($df = n_1 + n_2 - 2$ or Welch)	T-table	T-table
Paired T	T-distribution ($df = n - 1$)	T-table	T-table
F-Test	F-distribution	F-table	F-table
ANOVA	F-distribution	F-table	F-table
Chi-Squared	Chi-squared distribution	Chi-squared table	Chi-squared table
Shapiro-Wilk	Empirical	Not directly used	Empirical methods

Each test's critical value and p-value are derived based on its respective distribution.



This isn't very useful. You can remove the Shapiro-Wilks test from this next request.

1. Mathematically justify why we have each distribution for each test.
2. Mathematically explain what a critical value is and how they are calculated for each test.
3. Mathematically explain the calculation of a p-value for each test.

Additionally, explain what degrees of freedom and the significance level of a test are.

Preliminary Concepts

Before diving into each test, let's define **degrees of freedom** and **significance level** mathematically.

Degrees of Freedom (df)

Degrees of freedom (df) represent the number of independent values that can vary in a statistical calculation. It is generally given by:

$$df = \text{Total number of observations} - \text{Number of estimated parameters}$$

- In a **one-sample t-test**, $df = n - 1$ because we estimate one parameter (the mean).
- In a **two-sample t-test**, $df = n_1 + n_2 - 2$ because we estimate two means.
- In an **F-test (ANOVA or variance tests)**, the degrees of freedom are split between groups and within groups.

Significance Level (α)

The significance level is the probability of rejecting the null hypothesis when it is true (Type I error). It determines the critical region in hypothesis testing.

Mathematically, if the test statistic T follows distribution D , then the critical region is:

$$P(T > T_\alpha) = \alpha$$

for a one-tailed test, or

$$P(|T| > T_{\alpha/2}) = \alpha$$

for a two-tailed test.

1. One-Sample Z-Test

Distribution Justification

The sample mean follows a normal distribution:

$$\bar{x} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

By standardizing:

$$Z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} \sim N(0, 1)$$

since σ is known.

Critical Value Calculation

For a two-tailed test, the critical value $Z_{\alpha/2}$ satisfies:

$$P(|Z| > Z_{\alpha/2}) = \alpha$$

which we obtain from the standard normal table.

P-Value Calculation

For a two-tailed test:

$$p = 2P(Z > |Z_{\text{obs}}|)$$

2. Two-Sample Z-Test

Distribution Justification

$$Z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

Since $\bar{x}_1 - \bar{x}_2$ is normally distributed, Z follows a standard normal distribution.

Critical Value Calculation

Same as the one-sample Z-test.

P-Value Calculation

Same as the one-sample Z-test.

3. One-Sample T-Test

Distribution Justification

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

Since s^2 (sample variance) follows a **chi-squared distribution**:

$$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2(n-1)$$

it follows that:

$$t \sim t(n-1)$$

Critical Value Calculation

The critical value $t_{\alpha/2, df}$ satisfies:

$$P(|T| > t_{\alpha/2, df}) = \alpha$$

P-Value Calculation

For a two-tailed test:

$$p = 2P(T > |T_{\text{obs}}|)$$

4. Two-Sample T-Test

Distribution Justification

For equal variances:

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{s_p^2 \left(\frac{1}{n_1} + \frac{1}{n_2} \right)}}$$

where the pooled variance is:

$$s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

$$t \sim t(n_1 + n_2 - 2)$$

For unequal variances, Welch's approximation is used.

Critical Value Calculation

Same approach as the one-sample t-test, using $df = n_1 + n_2 - 2$.

P-Value Calculation

Same as the one-sample t-test.

5. Paired T-Test

Distribution Justification

Same as the one-sample t-test but applied to differences:

$$d_i = x_{1i} - x_{2i}, \quad \bar{d} = \frac{\sum d_i}{n}$$

$$t = \frac{\bar{d} - \mu_d}{s_d / \sqrt{n}} \sim t(n-1)$$

Critical Value Calculation

Same as the one-sample t-test.

P-Value Calculation

Same as the one-sample t-test.

6. F-Test for Variance Comparison

Distribution Justification

$$F = \frac{s_1^2}{s_2^2} \sim F(n_1 - 1, n_2 - 1)$$

where each variance follows a chi-squared distribution:

$$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2(n-1)$$

Critical Value Calculation

$$P(F > F_{\alpha, df_1, df_2}) = \alpha$$

from the F-table.

P-Value Calculation

$$p = P(F > F_{\text{obs}})$$

7. ANOVA (F-Test)

Distribution Justification

$$F = \frac{MSB}{MSW}$$

where:

$$MSB = \frac{SSB}{df_1}, \quad MSW = \frac{SSW}{df_2}$$

$$F \sim F(k-1, n-k)$$

Critical Value Calculation

From the F-table:

$$P(F > F_{\alpha, df_1, df_2}) = \alpha$$

P-Value Calculation

$$p = P(F > F_{\text{obs}})$$

8. Chi-Squared Test for Independence

Distribution Justification

$$\chi^2 = \sum \frac{(O_{ij} - E_{ij})^2}{E_{ij}}$$

where each term follows a chi-squared distribution, so:

$$\chi^2 \sim \chi^2(df)$$

with:

$$df = (r-1)(c-1)$$

Critical Value Calculation

From the chi-squared table:

$$P(\chi^2 > \chi_{\alpha, df}^2) = \alpha$$

P-Value Calculation

$$p = P(\chi^2 > \chi_{\text{obs}}^2)$$

Summary

Test	Distribution	Critical Values	P-Value
One-Sample Z	Normal ($N(0, 1)$)	$Z_{\alpha/2}$	(2P(Z >
Two-Sample Z	Normal ($N(0, 1)$)	$Z_{\alpha/2}$	(2P(Z >

Test	Distribution	Critical Values	P-Value
One-Sample T	$t(n - 1)$	$t_{\alpha/2, df}$	$(2P(T >$
Two-Sample T	$t(n_1 + n_2 - 2)$	$t_{\alpha/2, df}$	$(2P(T >$
Paired T	$t(n - 1)$	$t_{\alpha/2, df}$	$(2P(T >$
F-Test	$F(df_1, df_2)$	F_{α, df_1, df_2}	$P(F > F_{obs})$
ANOVA	$F(k - 1, n - k)$	F_{α, df_1, df_2}	$P(F > F_{obs})$
Chi-Squared	$\chi^2(df)$	$\chi_{\alpha, df}^2$	$P(\chi^2 > \chi_{obs}^2)$



Justify the distribution of each test statistics as much as you can.

Justification of the Distribution of Each Test Statistic

Below, I provide a detailed justification for why each test statistic follows its respective probability distribution.

1. One-Sample Z-Test

Test Statistic:

$$Z = \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}}$$

Justification for Normality:

1. The sample mean \bar{x} follows:

$$\bar{x} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

if the population is normally distributed, or approximately normal by the **Central Limit Theorem (CLT)** for large n .

2. Standardizing \bar{x} using the known population standard deviation σ :

$$Z = \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}}$$

results in a standard normal distribution:

$$Z \sim N(0, 1)$$

2. Two-Sample Z-Test

Test Statistic:

$$Z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

Justification for Normality:

1. Each sample mean follows:

$$\bar{x}_1 \sim N\left(\mu_1, \frac{\sigma_1^2}{n_1}\right), \quad \bar{x}_2 \sim N\left(\mu_2, \frac{\sigma_2^2}{n_2}\right)$$

(Normality holds by CLT if n_1, n_2 are large).

2. The difference of two independent normal distributions is also normal:

$$(\bar{x}_1 - \bar{x}_2) \sim N\left(\mu_1 - \mu_2, \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)$$

3. Standardizing gives:

$$Z \sim N(0, 1)$$

3. One-Sample T-Test

Test Statistic:

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

Justification for the t-Distribution:

1. If the population is normal:

$$\bar{x} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

2. The sample variance follows a **chi-squared distribution**:

$$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2(n-1)$$

because it is the sum of squared normal variables.

3. The ratio:

$$\frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

follows a **t-distribution** with $n - 1$ degrees of freedom:

$$t \sim t(n-1)$$

4. Two-Sample T-Test

Test Statistic:

$$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{s_p^2 \left(\frac{1}{n_1} + \frac{1}{n_2} \right)}}$$

Justification for the t-Distribution:

1. The variance estimator:

$$s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

is an **unbiased estimator** of the population variance.

2. The difference in sample means follows:

$$(\bar{x}_1 - \bar{x}_2) \sim N \left(\mu_1 - \mu_2, \frac{\sigma^2}{n_1} + \frac{\sigma^2}{n_2} \right)$$

but replacing σ^2 with s_p^2 introduces variability, leading to a **t-distribution** with $n_1 + n_2 - 2$ degrees of freedom.

For **Welch's t-test** (unequal variances), the degrees of freedom are approximated using:

$$df = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2} \right)^2}{\frac{\left(\frac{s_1^2}{n_1} \right)^2}{n_1 - 1} + \frac{\left(\frac{s_2^2}{n_2} \right)^2}{n_2 - 1}}$$

5. Paired T-Test

Test Statistic:

$$t = \frac{\bar{d} - \mu_d}{s_d / \sqrt{n}}$$

Justification for the t-Distribution:

- The paired t-test reduces to a one-sample t-test on the differences $d_i = x_{1i} - x_{2i}$.
- If the differences are normal, then:

$$\bar{d} \sim N \left(\mu_d, \frac{\sigma_d^2}{n} \right)$$

- The sample standard deviation of differences:

$$s_d^2 = \frac{1}{n - 1} \sum (d_i - \bar{d})^2$$

follows a chi-squared distribution.

- This leads to:

$$t \sim t(n-1)$$

6. F-Test for Variance Comparison

Test Statistic:

$$F = \frac{s_1^2}{s_2^2}$$

Justification for the F-Distribution:

- Each sample variance follows:

$$\frac{(n_1 - 1)s_1^2}{\sigma^2} \sim \chi^2(n_1 - 1), \quad \frac{(n_2 - 1)s_2^2}{\sigma^2} \sim \chi^2(n_2 - 1)$$

- The ratio of two chi-squared distributions divided by their degrees of freedom follows an **F-distribution**:

$$F \sim F(n_1 - 1, n_2 - 1)$$

7. ANOVA (F-Test)

Test Statistic:

$$F = \frac{\text{MSB}}{\text{MSW}}$$

Justification for the F-Distribution:

- The mean square between groups (MSB) and mean square within groups (MSW) are independent chi-squared distributed variables divided by their respective degrees of freedom:

$$\frac{\text{SSB}/(k-1)}{\sigma^2} \sim \chi^2(k-1), \quad \frac{\text{SSW}/(n-k)}{\sigma^2} \sim \chi^2(n-k)$$

- Their ratio follows:

$$F = \frac{\text{MSB}}{\text{MSW}} \sim F(k-1, n-k)$$

8. Chi-Squared Test for Independence

Test Statistic:

$$\chi^2 = \sum \frac{(O_{ij} - E_{ij})^2}{E_{ij}}$$

Justification for the Chi-Squared Distribution:

- The observed and expected counts follow a **multinomial distribution**.
- By Pearson's approximation, when sample sizes are large, the test statistic follows:

$$\chi^2 \sim \chi^2((r-1)(c-1))$$

where degrees of freedom are based on the number of independent constraints in the contingency table.

These justifications derive from fundamental probability theory, particularly properties of the **normal**, **chi-squared**, and **F-distributions**.