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procedural proficiency. In particular, mathematics teachers must know how to actively engage their students in the mathematical processes of problem solving, reasoning, representing, communicating, and making connections. The cycle of planning, implementing, and reflecting contributes both to students' learning and to teachers' continual improvement.

Summary of the Knowledge Standards

The Teaching and Learning Principles of *Principles and Standards for School Mathematics* set goals for a new vision of the mathematics classroom. Departing from traditional practices, teachers use their knowledge of mathematics, pedagogy, and their students to develop lessons that foster students' learning and engagement. They assume that all students can learn and use significant mathematics, and they craft lessons to involve and challenge all students. How teachers implement their knowledge in the classroom is examined in the next section.

Creating an environment that supports and encourages mathematical reasoning and fosters all students' competence with, and inclination toward, mathematics should be one of a teacher's central concerns.

Implementation

Teachers must implement their knowledge about the teaching and learning of mathematics by choosing challenging tasks and facilitating meaningful mathematical discourse within a healthy and supportive learning environment. Creating an environment that supports and encourages mathematical reasoning and fosters all students' competence with, and inclination toward, mathematics should be one of a teacher's central concerns. The nature of such a learning environment is shaped by the kinds of mathematical tasks and discourse in which students engage. The tasks must encourage students to reason about mathematical ideas, to make connections, and to formulate, grapple with, and solve problems. In practice, students' actual opportunities for learning depend, to a considerable degree, on the kind of discourse that the teacher orchestrates. The nature of the activity and talk in the classroom shapes each student's opportunities to learn about particular topics as well as to develop her or his abilities to reason and communicate about those topics (Henningsen and Stein 1997).

Standard 3: Worthwhile Mathematical Tasks

The teacher of mathematics should design learning experiences and pose tasks that are based on sound and significant mathematics and that—

- engage students' intellect;
- develop mathematical understandings and skills;
- stimulate students to make connections and develop a coherent framework for mathematical ideas;

- call for problem formulation, problem solving, and mathematical reasoning;
- promote communication about mathematics;
- represent mathematics as an ongoing human activity; and
- display sensitivity to, and draw on, students' diverse background experiences and dispositions.

Elaboration

The tasks and activities that teachers select are mechanisms for drawing students into the important mathematics that composes the curriculum. Worthwhile mathematical tasks are those that do not separate mathematical thinking from mathematical concepts or skills, that capture students' curiosity, and that invite students to speculate and to pursue their hunches. Many such tasks can be approached in more than one interesting and legitimate way; some have more than one reasonable solution. Such tasks, consequently, promote student communication about mathematics by facilitating classroom discourse. They require students to reason about different strategies and outcomes, weigh the pros and cons of alternatives, and pursue particular paths. Worthwhile mathematical tasks can be drawn from print or electronic resources, or they can be created by teachers. Tasks can be designed to incorporate a wide range of tools available for teaching mathematics, such as Web-based resources, computer software, manipulative materials, calculators, puzzles, or interactive electronic devices.

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Teachers should maintain a curricular perspective, considering the potential of a task to help students progress in their cumulative understanding in a particular domain and to make connections among ideas they have studied in the past and those they will encounter in the future. For example, elementary school students are often taught that adding fractions and adding decimals are two entirely unrelated procedures. But a task in which students are asked to represent the sum of two decimal numbers as a sum of fractions that correspond to tenths, hundredths, thousandths, and so on, might help students recognize the connections between the two processes with respect to the combining of like terms.

Rather than ask students to memorize mathematical vocabulary, worthwhile mathematical tasks can be embedded in meaningful contexts that help students see the need for definitions and terms as they learn new concepts. For example, a method for helping students develop an understanding of function concepts is to introduce mathematical terms associated with functions and provide examples that illustrate those terms. However, a more worthwhile alternative might be to select a problem that involves a particular functional relationship, that underscores the need to consider the domain and range of the function, and that requires students to identify the meaning (or meaningless nature) of the function's inverse in the problem context.

Another content consideration is to assess what the task conveys about what is entailed in doing mathematics. Some tasks, although they might deal nicely with the concepts and procedures at hand, involve students in simply producing right answers. Other tasks can require students to speculate, to pursue alternatives, and to face decisions about whether their approaches are valid. For example, one task might require students to find means, medians, and modes for given sets of data. Another might require them to decide whether to calculate means, medians, or modes as the best measures of central tendency, given particular sets of data and particular claims they would like to make about the data, then to calculate those statistics, and finally to explain and defend their decisions. Like the first task, the second would offer students the opportunity to practice finding means, medians, and modes. Only the second, however, conveys the important point that summarizing data involves making decisions related to the data and the purposes for which the analysis is being used.

Tasks should foster students' sense that mathematics is a changing and evolving domain, one in which ideas grow and develop over time and to which many cultural groups have contributed.

Tasks should foster students' sense that mathematics is a changing and evolving domain, one in which ideas grow and develop over time and to which many cultural groups have contributed. Drawing on the history of mathematics can help teachers portray that idea. For example, teachers may ask students to explore alternative numeration systems or to investigate non-Euclidean geometries. Fractions evolved out of the Egyptians' attempts to divide quantities, such as four things shared among ten people. That fact could provide the explicit basis for a teacher's approach to introducing fractions.

Still another content consideration centers on the development of appropriate skill and automaticity. Teachers must assess the extent to which skills play a role in the context of particular mathematical topics. A goal is to create contexts that foster skill development even as students engage in problem solving and reasoning. For example, elementary school students should develop rapid facility with addition and multiplication combinations. Rolling pairs of dice as part of an investigation of probability can simultaneously provide students with practice with addition. Trying to figure out how many ways thirty-six desks can be arranged in equal-sized groups—and whether more or fewer groupings are possible with thirty-six, thirty-seven, thirty-eight, thirty-nine, or forty desks—prompts students to quickly produce each number's factors. As they work on the equal-groupings problem, students have concurrent opportunities to practice multiplication facts and to develop a sense of what factors are. Further, the problem may provoke interesting questions: How many factors does a number have? Do larger numbers necessarily have more factors? Does a number exist that has more factors than 36? Even as students pursue such questions, they practice and use multiplication facts, for skill plays a role in problem solving at all levels. Teachers of algebra and geometry must similarly consider which skills are essential, and why, and seek ways to develop essential skills in contexts in which they matter. What do students need to memorize? How can that memorization be facilitated?

Other factors to consider are students' interests, dispositions, and experiences. Teachers should aim for tasks that are likely to engage their students' interests. Not always, however, should such concern for "interest" limit the teacher to tasks that relate to the familiar, everyday worlds of the students. For example, theoretical or fanciful tasks that challenge students intellectually are also interesting. When teachers work with groups of students for whom the notion of "argument" is uncomfortable or at variance with community norms of interaction, teachers must consider carefully the ways in which they help students engage in mathematical discourse.

Knowledge about ways in which students learn mathematics is a basis for appraising tasks. The mode of activity, the kind of thinking required, and the way in which students are led to explore the particular content all contribute to the learning opportunity afforded by the task. Knowing that students need opportunities to model concepts concretely and pictorially, for example, might lead a teacher to select a task that involves such representations. An awareness of common student confusions or misconceptions about a certain mathematical topic would lead a teacher to select tasks that engage students in exploring essential ideas that often underlie those confusions or misconceptions. Understanding that writing about ideas helps clarify and develop understandings would lend attractiveness to a task that requires students to write explanations. Teachers' views about how students learn mathematics should be guided by research as well as by their own experience. Just as teachers can learn more about students' thinking from the tasks they pose to students, so, too, can they gain insights into how students learn mathematics. To capitalize on the opportunity, teachers should deliberately select tasks that provide them with windows through which to view students' thinking.

Knowledge about ways in which students learn mathematics is a basis for appraising tasks.

Although worthwhile mathematical tasks are rich with the potential for engaging students, their potential cannot be reached without thoughtful placement in the curriculum and careful implementation in the classroom. Therefore, teachers must take as much care in implementing tasks as selecting them. To maintain student engagement, teachers must select tasks that are demanding enough to require high-level thinking and then must allow students to wrestle with the central ideas. Although teachers must choose when to guide, support, and assist students, they must also take care not reduce the demands of the tasks so much that the important mathematics is diluted and the value of the tasks is diminished (Stein, Grover, and Henningsen 1996).

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Standard 3: Worthwhile Mathematical Tasks

Vignette

In the first vignette, the teacher selects a task to meet her curricular goals and the needs of her students. The teacher analyzes two tasks on the basis of the tasks' potential to intellectually stimulate her students and involve them in the processes of reasoning and problem solving.

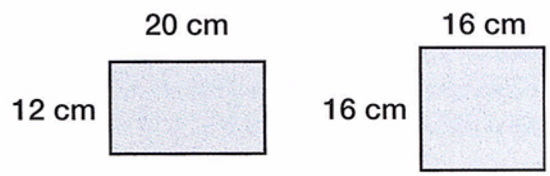
3.1—*Selecting Intellectually Stimulating Tasks*

Mrs. Jackson is thinking about how to help her students learn about perimeter and area. She realizes that learning about perimeter and area entails developing concepts, procedures, and skills. Students need to understand that the perimeter is the distance around a region, that area is the amount of surface inside a region, and that length and area are measured in two different units. Students need to realize that although perimeter and area are not, in general, functions of one other, they are determined by common dimensional measurements: two figures with the same perimeter may have different areas, and two figures with the same area may have different perimeters. Students also need to be able to determine the perimeter and the area of a given region. At the same time, they should connect what they learn about the measurements of perimeter and area with what they already know about other measures, such as measures of volume or weight.

Mrs. Jackson examines two tasks designed to help upper-elementary-grade students learn about perimeter and area. She wants to compare what each has to offer.

Task 1

Find the area and perimeter of each rectangle:



Mrs. Jackson decides that task 1 requires little more than correct application of formulas for perimeter and area. Nothing about the task requires students to make any conjectures about possible relationships between perimeter and area. This task is not likely to engage students intellectually; it does not entail reasoning or problem solving.

Task 2

Suppose you had 64 meters of fence with which you were going to build a pen for your large dog, Bones. What are some different sized and shaped pens you can make if you use all the fencing? What does the pen with the least play space look like? What is the biggest pen you can make—the one that allows Bones the most play space? Which pen size would be best for running?

Mrs. Jackson believes that task 2 can engage students intellectually because it challenges them to search for something. Although accessible to even young students, the problem is not immediately solvable. Neither is it clear how best to approach it. A question that students confront as they explore the problem is how to determine that they have indeed found the largest or the smallest play area. To justify an answer and to show that a problem is solved are essential components of mathematical reasoning and problem solving.

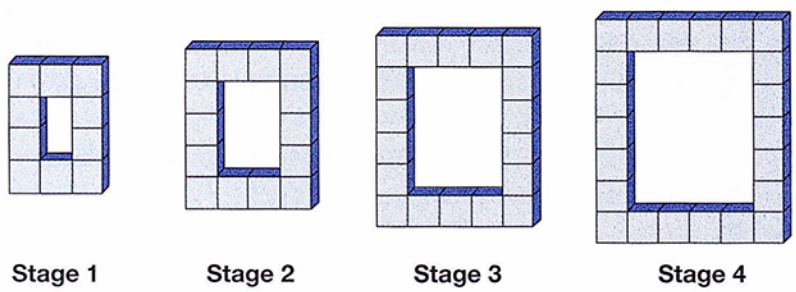
Vignette

In the second vignette, the teacher uses one problem as a basis for a series of class activities related to translating an arrangement of cubes into *a symbolic representation. Although the initial question the teacher poses can be solved by counting, he quickly makes use of the rich problem context to guide students to generate and give meaning to their own algebraic expressions.*

3.2—*Making Sense of Algebraic Expressions*

Juan Rodriguez is beginning a unit on algebraic expressions with his ninth-grade students. He wants to use a problem that he saw discussed at a professional conference to determine whether it will help his students make more sense of algebraic expressions and better understand how different expressions can be equivalent.

Mr. Rodriguez distributes a bag of connectable unit cubes to each group of three students. He shows a large-image projection of the following stages for everyone to see.



Mr. Rodriguez: In your groups, use your unit cubes to construct exact replicas of stage 1, stage 2, stage 3, and stage 4. Everyone at the table should check to make sure you have used the correct number of cubes in each stage.

The students enjoy the variety of activities that Mr. Rodriguez uses, so they quickly become engaged in the task. Mr. Rodriquez moves around the room to assure that students are focused on the task.

Mr. Rodriquez: On your mini white boards, please write the number of cubes that would be needed to construct the stage-5 figure. Raise your board so I can see your number.

The students discuss the question. Some students start building the stage-5 model while other students make calculations with paper and pencil. Eventually, all groups raise their boards.

Mr. Rodriguez: Sandra, can you tell us how your group got twenty-six?

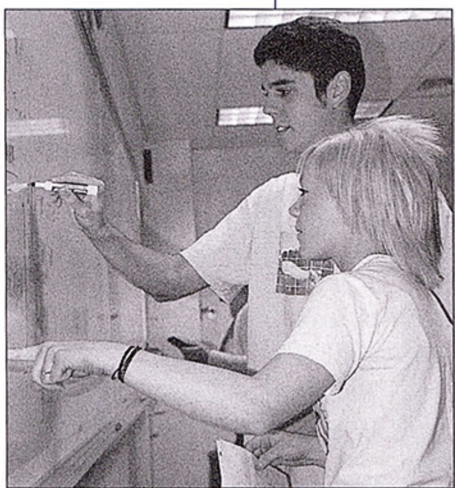
Sandra: We used our cubes to build the next stage, and then we counted how many cubes it took to make it.

Mr. Rodriguez: Did any other groups build the stage-5 model to answer the question?

Students in several groups raise hands.

Mr. Rodriguez: Roger, how did your group decide how many cubes are required to build stage 5?

Roger: Well, each stage is like a rectangle with the middle cut out. And the next one is one cube longer and one cube wider than the stage before. So we knew that this one would have to be sort of a frame that was seven cubes by eight cubes.



Photograph by Terry McGowan; all rights reserved

Mr. Rodriguez: Very well explained, Roger.

The teacher chooses to not take the time to have students share alternative solution strategies because he knows that those will come up in response to other questions he has planned.

Mr. Rodriguez: Now, everyone, go back to you groups and discuss how many cubes it would take to build stage 11 in this sequence. Please raise your white boards when you have a number.

Mr. Rodriguez: I can see that every group has come up with fifty. Amanda, how did your group get fifty?

Amanda: Well, we didn't have enough cubes to build it, so we had to think of another way. We started by thinking of the four corners, then we added the rest of the cubes on the sides. So we added 4 to 11 and 11 and 12 and 12.

Mr. Rodriguez: Okay. Did another group do it differently? Raul?

Raul: We noticed that the length of the frame was 2 more than the stage number and the width was 3 more than the stage number. So we added two lengths and two widths, or two 13s and two 14s. That gave us 54, but we realized that we counted the corners twice, so we subtracted 4.

After asking several other students to explain their method of solving the problem, Mr. Rodriguez poses the following extension: "How many cubes would it take to build a frame, as you all have started to call these, at stage *n?* When you have an expression, please record it on the front board."

Students record several different expressions, including the following:

$$2n + 2(n + 1) + 4$$

$$2(n + 2) + 2(n + 3) - 4$$

$$n + n + (n + 3) + (n + 3)$$

$$2n + 2(n + 3)$$

$$(n + 2) + (n + 2) + (n + 1) + (n + 1)$$

$$4n + 2 + 4$$

$$2(2n + 1) + 4$$

$$(n + 2)(n + 3) - (n)(n + 1)$$

Mr. Rodriquez is pleased with the variety of expressions that are generated. He asks the class, "These all seem different. How can they all be correct? Maybe we should start by having you explain where these came from so we can see if they make sense. I'm especially intrigued by the last one. Bonnie, can you tell us how your group came up with that?"

Bonnie: We thought about the frame as a rectangle taken away from the middle of a bigger rectangle. So the outer rectangle has dimensions $(n + 3)$ by $(n + 2)$, and the inner rectangle has dimensions (n) by $(n + 1)$. Then we just subtracted the inner area from the outer area.

After discussing the rest of the student-generated expressions, Mr. Rodriguez plans to give the class another pattern of blocks and ask them a similar series of questions. He hopes that they will move to a generalization more quickly so that he can begin to ask questions that will help students focus on the concept of equivalence of algebraic expressions.

When reflecting on this class session, the teacher concludes that the exploration was worthwhile because all the students were quickly engaged in the task, it required that students give meaning to variables and expressions, and a valuable class discussion ensued.

Closing Thoughts
Standard 3: Worthwhile Mathematical Tasks

To engage students in challenging mathematics, teachers need to use worthwhile mathematical tasks. Those tasks must be rich in terms of content and processes. Tasks that promote communication and connections can help students see and articulate the value and beauty of mathematics. Teachers can enhance the value of existing materials by tailoring them to needs and interests of their students. By doing so, teachers can promote both motivation and equity in their classrooms.

Tasks that promote communication and connections can help students see and articulate the value and beauty of mathematics.

Standard 4: Learning Environment

The teacher of mathematics should create a learning environment that provides—

- the time necessary to explore sound mathematics and deal with significant ideas and problems;