"What do the rest of you think about that?" inquires the teacher.

Several children give explanations in support of Frankie's solution. "I think that does make sense," says one girl, "but I had another solution. I think the answer is 1 plus 1/2 plus 1/4."

"I don't understand," Ms. Morgan says. "Could you show what you mean?"

Ms. Morgan encourages her students to take risks by bringing up different ideas. But raising questions is not enough. She also reminds students that they are expected to clarify and justify their ideas.

Closing Thoughts
Standard 4: Learning Environment

To capitalize on the value of worthwhile mathematical tasks, teachers and students must explore those tasks within a classroom environment that fosters intellectual growth and development. In such an environment, an objective of both teachers and students is to use mathematical tools and skills to make sense of problems, patterns, and contradictions. Participants use appropriate technology in such an environment to foster insight into mathematical situations. Students work independently or collaboratively to develop skills, make conjectures, and develop arguments within a mathematical community that values the contributions of all participants and defers to the authority of sound reasoning in the search for mathematical truth.

Standard 5: Discourse

The teacher of mathematics should orchestrate discourse by—

- posing questions and tasks that elicit, engage, and challenge each student's thinking;
- listening carefully to students' ideas and deciding what to pursue in depth from among the ideas that students generate during a discussion;
- asking students to clarify and justify their ideas orally and in writing and by accepting a variety of presentation modes;
- deciding when and how to attach mathematical notation and language to students' ideas;
- encouraging and accepting the use of multiple representations;
- making available tools for exploration and analysis;
- deciding when to provide information, when to clarify an issue, when to model, when to lead, and when to let students wrestle with a difficulty; and
- monitoring students' participation in discussions and deciding when and how to encourage each student to participate.

Elaboration

The discourse of a classroom—the ways of representing, thinking, talking, agreeing, and disagreeing—is central to what and how students learn about mathematics.

The Process Standards, most particularly the Communication Standard and the Proof and Reasoning Standard, make explicit reference to the role of discourse in the mathematics classroom (NCTM 2000). The discourse of a classroom—the ways of representing, thinking, talking, agreeing, and disagreeing—is central to what and how students learn about mathematics. Discourse represents both what the ideas entail and the ways ideas are exchanged: Who talks? About what? In what ways? What do people write? What do they record, and why? What questions are important? How do ideas change? Whose ideas and ways of thinking are valued? Who determines when to end a discussion? The discourse is shaped by the tasks in which students engage and the nature of the learning environment. The teacher's role is to initiate and orchestrate discourse and to use it skillfully to foster students' learning.

Like a piece of music, the classroom discourse has themes that are synthesized into a whole that has meaning. The teacher has a central role in orchestrating oral and written discourse in ways that contribute to students' understanding of mathematics. Students must be given opportunities to engage in making conjectures, to share their ideas and understandings, to propose approaches and solutions to problems, and to argue about the validity of particular claims; they must recognize that mathematical reasoning and evidence are the bases for discourse (NCTM 2000).

To effectively orchestrate mathematical discourse, teachers must do more listening, and students must do more reasoning.

To effectively orchestrate mathematical discourse, teachers must do more listening, and students must do more reasoning. Because many more ideas often come up than are fruitful to pursue at a given moment, teachers must filter and direct the students' explorations by picking up on some points and by leaving others behind. Doing so prevents student discourse from becoming too diffuse and unfocused. Such decisions depend on teachers' understandings of mathematics and of their students. In particular, teachers must make judgments about when students can figure things out individually or collectively and when they require additional input.

Knowledge of mathematics, knowledge of the curriculum, and knowledge of students should guide the teacher's decisions about the nature and the path of the discourse. Beyond asking clarifying or provocative questions, teachers should also, at times, provide information and lead students. Decisions about when to let students work to make sense of an idea or a problem without direct teacher input, when to ask leading questions, and when to tell students something directly are crucial to orchestrating productive mathematical discourse the classroom. Above all, the discourse should be focused on making sense of mathematical ideas and on using mathematical ideas sensibly in setting up and solving problems (Hiebert et al. 1997).

But how can teachers both stimulate and manage classroom discourse? Here are several suggestions:

1. Provoke students' reasoning about mathematics. Teachers do so through the tasks they provide and the questions they ask. For example, teachers should

regularly follow students' statements with "Why?" or "How do you know?" Doing so consistently, irrespective of the correctness of students' statements, is an important part of establishing a discourse centered on mathematical reasoning. In particular, when students are allowed to examine and critique incorrect solutions or strategies, counterexamples and logical inconsistencies can naturally surface. This process of analyzing solutions instead of relying on teachers to validate them can enhance students' abilities to think critically from a mathematical perspective.

Tasks and questions that stimulate discussion not only encourage active participation by students but also provide the teacher with ongoing assessment information. Cultivating a tone of interest when asking a student to explain or elaborate on an idea helps establish norms of civility and respect rather than criticism and doubt. The teacher must create an environment in which everyone's thinking is respected and in which reasoning and arguing about mathematical meanings is the norm.

2. Encourage the use of a variety of representations as well as explorations of how various representations are alike and different. Require students to write explanations for their solutions and to provide justifications for their ideas. Students should learn to verify, revise, and discard claims on the basis of mathematical evidence. Although students should learn to use and appreciate the conventional symbols that facilitate mathematical communication, teachers must value and encourage the use of a variety of tools and representations as students are learning to communicate their ideas. Various means for communicating about mathematics should be accepted, including drawings, diagrams, invented symbols, and analogies. The teacher should introduce conventional notation when students are ready to formalize their ideas and in a manner that helps students understand that using agreed-on terms and notations helps promote clear communication.

   As described in the Technology Principle, teachers should also help students learn to use calculators, computers, Internet resources, and other technological devices as tools for mathematical discourse. Given the range of mathematical tools available, teachers should often allow and encourage students to select the means they find most useful for working on or discussing a particular mathematical problem.

3. Monitor and organize students' participation. Teachers must be committed to engaging every student in contributing to the overall thinking of the class. Teachers must judge when students should work and talk in small groups and when the whole group is the most useful context. They must make sensitive decisions about how opportunities to speak are shared in the large group—for example, whom to call on, when, and whether to call on particular students who do not volunteer. Substantively, if discourse is to focus on making sense of mathematics, teachers must refrain from calling only on students who seem to have right answers or valid ideas to allow a broader spectrum of thinking to be explored in the discourse. By modeling respect for students' thinking and conveying the assumption that students make sense, teachers can encourage

students to participate within a norm that expects group members to justify their ideas.

Engaging every student in the discourse of the class requires considerable skill as well as an appreciation of, and respect for, students' diversity.

To facilitate learning by all students, teachers must also be perceptive and skillful in analyzing the culture of the classroom, looking out for patterns of inequality, dominance, and low expectations that are primary causes of non-participation by many students. Engaging every student in the discourse of the class requires considerable skill as well as an appreciation of, and respect for, students' diversity.

4. Encourage students to talk with one another as well as in response to the teacher. Whether working in small or large groups, students should be the audience for one another's comments—that is, they should speak to one another, aiming to convince or to question their peers. When the teacher talks most, the flow of ideas and knowledge is primarily from teacher to student. When students make public conjectures and reason with others about mathematics, ideas and knowledge are developed collaboratively, revealing mathematics as constructed by human beings within an intellectual community. Students, who are accustomed to the teacher's doing most of the talking while they remain passive, need guidance and encouragement to participate actively in the discourse of a collaborative community. Some students, particularly those who have been successful in more traditional mathematics classrooms, may be resistant to talking, writing, and reasoning together about mathematics. Time and patience are required to cultivate a student-centered environment for discourse. Eventually, the payoff in student participation and learning is worth the investment.

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Standard 5: Discourse

Vignette

In the first vignette, the teacher facilitates a discussion about counting with young children. The choice of task creates an opportunity for reasoning and problem solving. The teacher's consistent urging of students to explain and to justify their answers invites many students into the conversation and encourages the sharing of a variety of solution methods.

5.1—*Only the Nose Knows, but the Children Can Reason!*

Ms. Nakamura has done a lot of number work with her kindergarten class this year, and she is pleased with the results. Now, near the end of the year, the class has been investigating patterns in the number of various body parts in the classroom—how many noses or eyes, for example, are present among the children in the class.

Earlier that week, each child had made a nose out of clay. Ms. Nakamura opens the discussion by revisiting that project. She asks, "And how many noses did we make?"

Becky (points to her nostrils): Two of these.

Teacher: But how many actual noses?

Anne: Twenty-nine.

Teacher: Why? Why were there twenty-nine noses?

Adam: Because every kid in the class made one clay nose, and that is the same number as kids in the class.

Teacher (pointing to her nostrils): Now Becky just said—remember what these are called?

Children: Nostrils!

Teacher: So were there twenty-nine nostrils?

Pat: No, there were more.

Gwen: Fifty-eight! We had fifty-eight nostrils!

Teacher: Why fifty-eight?

Gwen: I counted.

Felice: If we had thirty kids, we would be sixty. So it is fifty-nine 'cause it should be one less.

Teacher: Can you explain that again?

The teacher probes Felice's answer even though her approach goes beyond what many of the children are trying to do at this point.

Felice: It's fifty-nine because we don't have thirty kids, we have twenty-nine, so it is one less than sixty.

Teacher: What does anyone else think?

Adam: I think it is fifty-eight. Each kid has two nostrils. So if sixty would be for thirty kids, then it has to be two less: fifty-eight.

The teacher solicits other students' reactions instead of showing them the right answer. Her tone of voice and her questions show the students that she values their thinking.

Lawrence: But Felice says thirty kids makes sixty ...

Felice: No! Adam makes sense. Fifty-eight.

The teacher moves on, asking, "What else do you think we have on our bodies that would be more than twenty-nine?"

The teacher's question challenges students to think it is open-ended; more than one right answer exists.

Graham: More than twenty-nine fingers.

Teacher: More than twenty-nine fingers? Why do you think so?

Graham: Because each kid, we have ten fingers.

Ricky: More than twenty-nine shoes.

Teacher: More than twenty-nine shoes. And what are those shoes covering?

Ricky: Your feet.

Sarah: Ears.

Beth: More than twenty-nine legs.

Ms. Nakamura tells the children, "You did some good thinking today!" The teacher chooses to comment on the children's thinking instead of their behavior.

Ms. Nakamura tells the children that they now are to work on a picture: "Choose some body part, and draw a picture of how many of those we have in our class and how you know that." She directs the children back to their tables, where she has laid out paper and cans of crayons.

Vignette

In the following vignette, the teacher encourages the use of multiple representations to help students construct understandings. The students use objects and symbols to help convey their meaning. The teacher eventually introduces standard notation and helps students make connections between their own and other representations.

5.2—*Making the Transition from Student-Invented to Standard Representations*

Mr. Johnson has presented his first-grade class with several pairs of numbers and asked them to decide which number in each pair is greater and to justify their responses. He has also been encouraging them to find ways to write their comparisons. The students know that the teacher expects not just answers but also reasons.

Ben: I think 5 is greater than 3 because [walks to the board and sticks five magnets up and then carefully sticks three magnets in another row]).

Mr. Johnson asks whether that explanation makes sense to other people. The children nod. He asks whether anyone wants to show how you would write Ben's idea down. The teacher encourages students to use symbols to represent and communicate about ideas.

Kevin, up at the board, writes this expression:

$$5 \rightarrow 3$$

Next, Betsy writes the following:

$$5 \quad 3$$

Mr. Johnson: Can you explain what you were thinking? Kevin?

Kevin explains that his arrow shows that 5 is more than 3 because the bigger number "can point at" the smaller one.

Mr. Johnson asks Betsy to explain hers, and she says that she thinks you should just circle the smaller one.

The teacher accepts more than one way of representing the idea of comparing numbers by using symbols; both are nonstandard but sensible. However, Mr. Johnson challenges his students by posing a question that requires students to invent a means of recording an idea.

Photograph courtesy of Deborah A. Niezgoda and Patricia S. Moyer-Packenham; all rights reserved

Mr. Johnson: What if the two numbers you were comparing were 6 and 6? What would you do? How would you use your symbols for comparing numbers to write that?

Several seconds pass. The teacher gives students time to think before responding. He doesn't repeat the question or call on children; he is patient and silent. Eventually, Ruth shoots her hand in the air. Several others also have their hands up.

Ruth: You could draw an arrow to both of them.

Annie: You could circle both of them because they are the same.

Jimmy: You shouldn't mark either one, either way. They are not greater or less. They are the same.

Mr. Johnson nods at their suggestions. He writes an equals sign (=) on the board and explains that this is a symbol that people have invented for the ideas the children have been talking about.

The teacher connects the students' approaches and reasoning with the conventional notation. Because the students have thought about what it means for two numbers to be equal, they are ready to learn how that relationship is conventionally represented. In this situation, the notation follows the development of the concept in a meaningful context.

Rashida: That's like what Ruth said.

Ruth beams, and Annie calls out, "It's like mine, too."

Vignette

In the third vignette, a teacher introduces a collaborative project. The teacher monitors small-group discussion and planning. He offers advice and resources when appropriate.

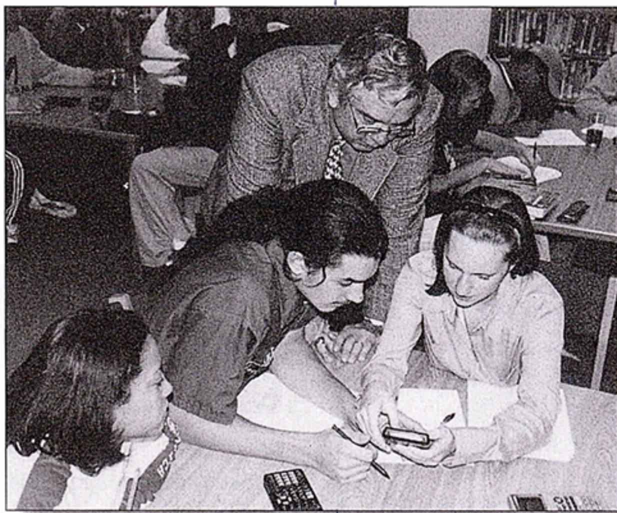
5.3—*Letting the Discourse Happen: Monitoring Collaborative Groups*

Mr. Cohen's class of high school students is working in small groups on projects that involve collecting, organizing, and interpreting data. The teacher has posed a task that gives the students an opportunity to develop their understanding of *sample* and *population*. The task is also intended to help students extend their ability to use statistics to reason about real-world situations.

The class first discussed possibilities for projects. They identified questions that they wanted to pursue, such as finding the average number of hours per week that high school students work.

In one group, a student had recently read a newspaper article on changes in the popularity of first names since 1925. The group has decided to investigate the most common boys' and girls' first names among students their age in their community. Are Michael, James, and Robert still the most common boys' names, as they were in 1925, 1950, and 1975, respectively? They are curious about what has happened with girls' names, because, according to the article, girls' popular names seem to change more often. They also wonder how their community's ethnic diversity affects the pattern of names.

Mr. Cohen moves around the room to the different groups. He stops to listen, offer suggestions, and verify that the group members are listening to one another and working together. The teacher is providing time for students to grapple with the data-collection aspects of their projects. The group that is working on the names study has decided to sample the high school population in the city and is discussing the best way to go about doing so. They intend to compare their results with national data available from the Social Security Administration Web site, a data source that was provided in the newspaper article.



Photograph courtesy of Joseph Dakoske; all rights reserved

John: Let's choose three of the high schools and then write to them and ask for a list of the students enrolled in the school. We can sample names at random from those lists.

Jenny: But how will we pick the three schools? And why is three a good number to pick?

John: It seemed like enough out of all the schools in our city if we were careful to include one of the schools that has more kids from different backgrounds, because we want to make sure our sample has lots of different kinds of names, just like there are around here.

Anna: I think we should try to figure out about how many high school kids there are in the whole city and then pick a size for our sample based on that.

John (nodding): I guess that makes sense. How are we going to figure that out, though?

Maria: And then how big would our sample have to be to be big enough? We want to be pretty sure that our sample tells us something about all the kids in high school here.

Mr. Cohen is standing by the group. These students seem to have developed the disposition to question one another, and they seem accustomed to thinking through problems together.

These students seem to have developed the disposition to question one another, and they seem accustomed to thinking through problems together.

Mr. Cohen tells the students that their discussion so far is productive, that they are dealing with some important questions for their project. He suggests a source that might help them think about the question of how many students they need to have in their sample. He also tells them that the school administration office would have a list of all the high schools and how many students attend each of them. He asks if they would like him to call and ask for that list. They say that they would. He asks what they are going to do once they get the list.

Closing Thoughts
Standard 5: Discourse

Although some level of communication occurs in every mathematics classroom, the type and the amount of participation in that communication are varied. In one-way communication from teacher to students, students have very little opportunity to

Careful monitoring of student discourse and posing questions that challenge students to refine and reorganize their ideas improve the quality of the discourse as well as students' developing understandings.

grapple with mathematical ideas within an intellectual community. When discourse patterns include student-to-student discourse and student-to-teacher discourse, students find opportunities to translate their thoughts into verbal, symbolic, and graphical representations. By using those representations, students develop facility with them and attach meaning to them. Careful monitoring of student discourse and posing questions that challenge students to refine and reorganize their ideas improve the quality of the discourse as well as students' developing understandings.

Summary of the Implementation Standards

Implementation of teachers' knowledge is evident in the ways teachers choose tasks, the means by which they create a supportive and challenging environment for learning, and the tools they use to orchestrate discourse in the classroom. Because the teacher is responsible for shaping and directing students' activities so that they have opportunities to engage meaningfully in mathematics, the tasks in which students engage must encourage them to reason about mathematical problems. By expecting students to participate, listen respectfully to one another, present their ideas, and pose questions to the teacher and to peers, the teacher establishes an environment that nurtures the learning of mathematical processes and concepts and the development of skills.

The discourse of the mathematics class reflects messages about what it means to know mathematics, what makes something true or reasonable, and what doing mathematics entails. It is central to both what students learn about mathematics and how they learn it. Therefore, the discourse of the mathematics class should be founded on mathematical ways of knowing and ways of communicating. Although a teacher may be more of a "guide on the side" than a "sage on the stage," the teacher is the central element in fostering worthwhile mathematical discourse within the classroom community.

Teachers' skills in developing and integrating the tasks, discourse, and environment in ways that promote students' learning are enhanced through the thoughtful analysis of their instruction. That process of analysis is the focus of the next, and final, group of standards in this chapter.

Analysis

A central question to which teachers must be prepared to respond is "How well are the tasks, discourse, and learning environment working to foster the development of students' mathematical proficiency and understanding?"

Trying to understand as much as possible about the various elements that affect the learning of each student is essential to good teaching.

Trying to understand as much as possible about the various elements that affect the learning of each student is essential to good teaching. Teachers must monitor classroom life using a variety of strategies and focusing on a broad array of dimensions of mathematical competence, as described in the Assessment Principle of *Principles* *and Standards for Schools Mathematics* (NCTM 2000). The information gained from formative assessment should guide instruction. What do students seem to understand well, and what do they appear to understand only partially? What connections do they seem to be making? What dispositions toward mathematics do they convey? How does the class work together as a learning community making sense of mathematics? What teachers learn from their analysis should be a primary source of information for planning and improving instruction.

Standard 6: Reflection on Student Learning

The teacher of mathematics should engage in ongoing analysis of students' learning by—

- observing, listening to, and gathering information about students to assess what they are learning

so as to—

- ensure that every student is learning sound and significant mathematics and is developing a positive disposition toward mathematics;
- challenge and extend students' ideas;
- adapt or change activities while teaching;
- describe and comment on each student's learning to parents and administrators; and
- provide regular feedback to the students themselves.

Elaboration

Assessing students and analyzing instruction are fundamentally interconnected. Mathematics teachers should monitor students' learning on an ongoing basis to assess and adjust their teaching. Observing and listening to students during class can help teachers, on the spot, tailor their questions or tasks to provoke and extend students' thinking and understanding. Students' dispositions toward mathematics—their confidence, interest, enjoyment, and perseverance—constitute another important dimension for teachers to monitor. Teachers have the responsibility of describing and commenting on students' learning to administrators, to parents, and to the students themselves so as to guide each student to a better understanding of his or her personal learning style.

Teachers must assess the skills, knowledge, and conceptual and procedural understanding of their students. They must also assess the development of students' ability to reason mathematically—to make conjectures, to justify and revise claims on the basis of mathematical evidence, and to analyze and solve problems. Paper-and-pencil tests, although one example of a useful means for judging certain aspects of students'