Chapter 1

Standards for Teaching and Learning Mathematics

Overview

This section presents seven Standards for the teaching of mathematics organized under three categories: knowledge, implementation, and analysis.

Knowledge

1. Knowledge of Mathematics and General Pedagogy
2. Knowledge of Student Mathematical Learning

Implementation

3. Worthwhile Mathematical Tasks
4. Learning Environment
5. Discourse

Analysis

6. Reflection on Student Learning
7. Reflection on Teaching Practice

Introduction

Principles and Standards for School Mathematics represents NCTM's vision of school mathematics. That vision is designed to guide educators and others involved in the support of teaching and learning as they strive to improve classrooms and schools. The Principles of Equity, Curriculum, Teaching, Learning, Assessment, and Technology are the cornerstones of high-quality mathematics education. Those Principles describe high expectations and support for all students; a coherent, focused curriculum; teaching based on knowledge of students and content; meaningful learning that is actively constructed; multiple forms and purposes of assessment; and the appropriate use of technology to enhance understanding (NCTM 2000).

The vision for school mathematics that is articulated in *Principles and Standards* requires changes in what mathematics is taught and how it is taught. Teachers and students have

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different roles and different notions about what it means to know and to do mathematics. Teachers expect students to encounter, develop, and use mathematical ideas and skills in the context of genuine problems and situations. In so doing, students develop the ability to use a variety of resources and tools, such as calculators and computers as well as concrete, pictorial, and metaphorical models. They know and are able to choose appropriate methods of computation, including estimation, mental calculation, and the use of technology. As they explore and solve problems, they engage in conjecture and argument.

The purpose of this chapter is to sharpen and expand the images of teaching and learning mathematics to elaborate the vision set forth in *Principles and Standards for School Mathematics*.

Seven Standards represent the core dimensions of teaching and learning mathematics. Those Standards are organized under three headings—knowledge, implementation, and analysis—that represent major arenas of teachers' work that are central to defining what occurs in mathematics classrooms.

Effective mathematics teaching depends on a deep knowledge of mathematics. Teachers need to understand the big ideas of mathematics and be able to represent mathematics as a coherent and connected enterprise.

- *Knowledge* of teaching, of mathematics, and of students is an essential aspect of what a teacher needs to know to be successful. Effective mathematics teaching depends on a deep knowledge of mathematics. Teachers need to understand the big ideas of mathematics and be able to represent mathematics as a coherent and connected enterprise (Ball, Lubienski, and Mewborn 2001; Hill, Rowan, and Ball 2005; Ma 1999; Schifter, Russell, and Bastable 1999). In addition, teachers must have sound pedagogical knowledge and a grounded understanding of students as learners. Teachers use that knowledge to create learning communities that enable students to build a conceptual understanding of, and procedural proficiency with, mathematics.
- *Implementation* of learning activities within the classroom requires a teacher to choose worthwhile mathematical tasks, to establish a supportive and challenging environment, and to promote mathematical discourse among all members of the learning community. The learning environment results from the unique interplay of intellectual, social, and physical characteristics that shape the ways of knowing and working that are encouraged and expected in the classroom. Worthwhile mathematical tasks challenge students to make sense of both the contexts and the mathematics embedded in the tasks. The discourse of the learning community refers to the ways of representing, thinking, talking, and agreeing and disagreeing that teachers and students use as they engage in mathematical thinking and learning. Teachers, through the ways in which they orchestrate discourse, convey messages about whose knowledge and whose ways of thinking and knowing are valued, who is considered able to contribute, and who has status in the group. Discourse also reflects classroom values, such as what makes an answer right, what counts as legitimate mathematical activity, and what the standards are for an argument to be considered convincing.

- *Analysis* refers to the systematic reflection in which teachers engage. It entails the ongoing monitoring of classroom life: How well are the tasks, discourse, and environment fostering the development of every student's mathematical proficiency and understanding? Such systematic reflection may also involve colleagues' working together to analyze and improve teaching practice. Further, teachers use information garnered from both formative and summative assessment to guide instructional decisions. Through that process, teachers examine relationships between what they and their students are doing and what students are learning.

In deciding how to present and elaborate the ideas underlying each of the seven Standards, we confronted two basic challenges. First, teaching is an integrated activity. Although we have described elements of teaching as falling into three categories—knowledge, implementation, and analysis—they are, in fact, interwoven and interdependent. The quality of the classroom discourse, for example, is both a function of, and an influence on, the teacher's knowledge of mathematics. Similarly, mathematical tasks are shaped by the teacher's knowledge of her students and her reflections on the use of those tasks in other situations. Our second challenge was that professional standards for mathematics teaching should represent values about what contributes to good practice without prescribing it. Such standards should offer a vision, not a recipe.

The format of this chapter grew out of consideration of those challenges. Because teaching is an integrated activity and because we wanted to provide concrete images of a vision, we have chosen to use illustrative vignettes of classroom teaching and learning. The statement of each of the seven Standards is first elaborated with an explanation of its main ideas. Each explanation is then followed with one or more illuminating scenarios drawn from transcripts, observations, and experiences in a wide variety of real classrooms. The vignettes were selected to reflect a range of teaching styles, classroom contexts, mathematical topics, and grade levels. The vignettes were gathered from classrooms with students of diverse cultural, linguistic, and socioeconomic backgrounds and include examples of teachers facing problems as well as instances of accomplished practice.

Assumptions

The seven Standards for teaching are based on four assumptions about the practice of mathematics teaching.

1. *Principles and Standards for School Mathematics* furnishes the basis for a curriculum in which problem solving, reasoning and proof, communication, connections, and representation are central. Teachers must help all students develop conceptual and procedural understandings of number and operations, algebra, geometry, measurement, and data analysis and probability. They must engage all students in formulating and solving a wide variety of problems, making conjectures and constructing arguments, validating solutions,

and evaluating the reasonableness of mathematical claims. Teachers should make use of appropriate technologies to assist students in such endeavors. Teachers must also foster the disposition to use and engage in mathematics, an appreciation of its beauty and utility, and a tolerance for getting stuck or side-tracked. Teachers must help students realize that mathematical thinking may involve dead ends and detours, all the while encouraging them to persevere when confronted with a puzzling problem and to develop the self-confidence and interest to do so.

***What* students learn is fundamentally connected with *how* they learn it.**

2. *What* students learn is fundamentally connected with *how* they learn it. Students' opportunities to learn mathematics emerge from the setting and the kinds of tasks and discourse in which they participate. What students learn about particular concepts and procedures as well as about mathematical thinking depends on the ways in which they engage in mathematical activity with and without technology in their classrooms. Their disposition toward mathematics is also shaped by such experiences. Consequently, the goal of creating high-quality learning experiences for all students requires careful attention to pedagogy as well as to curriculum.
3. All students can learn to think mathematically. The vision described in *Principles and Standards for School Mathematics,* and as articulated in the Equity Principle, is a vision that applies to all students. Although not all students learn in the same ways, all students should be expected to engage in mathematical processes to develop the tools to make sense of mathematical ideas. More specifically, the activities of making conjectures, arguing about mathematics using mathematical evidence, formulating and solving problems—even perplexing ones—are not just for some group of students thought to be "bright" or "mathematically able" (Trentacosta and Kenney 1997). Every student can and should learn to reason and solve problems, to make connections across a rich web of topics and experiences, and to represent and communicate mathematical ideas with and without technology. In fact, engaging in those activities, or thinking mathematically, is the essence of doing mathematics.
4. Teaching is a complex practice. Teaching is composed of elements that interact with and reinforce one another (Stigler and Hiebert 1999). In particular, teaching mathematics draws on knowledge from several domains: knowledge of mathematics, of diverse learners, of how students learn mathematics, and of the contexts of classroom, school, and society. Such knowledge is general but not superficial. However, teachers must also take into account that teaching is context-specific. Theoretical knowledge about adolescent development, for instance, can only, in part, influence a decision about particular students learning a particular mathematical concept in a particular context. Teachers often find themselves balancing multiple goals and considerations as they weave together knowledge to decide how to respond to a student's question, how to represent a given mathematical idea, how long to pursue the discussion of a problem, or how to make appropriate use of available technologies to develop the richness of an investigation. Making appropriate decisions depends on a variety of factors that cannot be determined in the abstract or be governed by rules of thumb.

The challenge of teaching mathematics well depends on a host of considerations and understandings. Good teaching demands that teachers thoughtfully apply best available knowledge about mathematics, learning, and teaching to the particular contexts of their work. The Standards for teaching mathematics are designed to help guide the processes of such reasoning, highlighting issues that are crucial in creating the kind of teaching practice that supports the goals of *Principles and Standards for School Mathematics*. This chapter promulgates themes and values but does not—indeed, could not—prescribe "right" practice.

Knowledge

The Teaching Principle from *Principles and Standards for School Mathematics* elaborates the complexity of knowledge that an effective teacher needs: mathematical content, pedagogy, assessment strategies, and an understanding of students as learners. Teachers need a sound knowledge of the concepts, skills, and reasoning processes of mathematics to construct and achieve short- and long-term curricular goals. Teachers must develop a repertoire of strategies and pedagogical knowledge to guide instructional decisions. That repertoire must be coupled with a sound knowledge of students and how to further students' learning. Knowing that students make sense of mathematics in differing ways, teachers use their knowledge to create lessons that address, build on, and extend previous knowledge. No single "right way" exists to teach all mathematical topics in all situations; effective teachers balance their knowledge of mathematics, knowledge of pedagogical strategies, and knowledge of students to help students become independent mathematical thinkers.

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Standard 1: Knowledge of Mathematics and General Pedagogy

Teachers of mathematics should have a deep knowledge of—

- sound and significant mathematics,
- theories of student intellectual development across the spectrum of diverse learners,
- modes of instruction and assessment, and
- effective communication and motivational strategies.

Elaboration

A goal of mathematics instruction is to enhance all students' understandings of both the concepts and the procedures of mathematics. To involve students in work that helps them deepen and connect their knowledge, teachers need specialized, content-specific knowledge for teaching mathematics. Specifically, teachers must themselves be experienced and highly skilled in the processes of problem solving, proof and

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reasoning, communications, connections, and representation. To help students understand the connections within and across content domains and between mathematics and other disciplines, teachers must have a wide and deep knowledge of mathematics centered on the school mathematics they teach (Ball and Cohen 1999). Such knowledge includes—

> mathematical facts, concepts, procedures, and the relationships among them; knowledge of the ways that mathematical ideas can be represented; and knowledge of mathematics as a discipline—in particular, how mathematical knowledge is produced, the nature of discourse in mathematics, and the norms and standards of evidence that guide argument and proof.
>
> (National Research Council 2001, p. 371)

Equally important is how that knowledge is held by the teacher. Is it a collection of disconnected facts and algorithms, or is it a coherent, interconnected set of concepts that underlie and explain those facts (Ma 1999)?

Furthermore, teachers' mathematical knowledge should be framed within an understanding of human intellectual development. In planning for instruction and assessment, teachers must consider what they know about their students as well as what they know more generally about students from psychological, cultural, sociological, and political perspectives. Specifically, teachers should be well informed about issues of equity so they can ensure that their lessons contribute to a positive learning experience for all. The Equity Principle reminds us that all students can learn and do mathematics, that each one is worthy of being challenged intellectually, and that reasonable and appropriate accommodations should be made as needed.

But setting high expectations for all students is not enough. To elicit, explore, and critique students' mathematical thinking requires careful planning. Several studies have identified important ways in which teachers should attend to students' thinking in their planning and teaching by (1) understanding the mathematical concepts that will be developing during a lesson (e.g., Borko and Livingston 1989; Fernandez and Yoshida 2004; Lampert 2001; Leinhardt 1993; Leinhardt and Steele 2005; Livingston and Borko, 1990; Schoenfeld 1998; Schoenfeld, Minstrell, and van Zee 2000; Stigler and Hiebert 1999); (2) anticipating the variety of strategies students may use in solving a problem as well as the misconceptions or difficulties students may have (e.g., Borko and Livingston 1989; Fernandez and Yoshida 2004; Lampert 2001; Leinhardt 1993; Livingston and Borko 1990; Schoenfeld 1998; Schoenfeld, Minstrell, and van Zee 2000; Stigler and Hiebert 1999); and (3) asking questions to elicit students' thinking and advance students' understanding (e.g., Fernandez and Yoshida 2004; Schoenfeld 1998; Stigler and Hiebert, 1999). Attending to student thinking may also take a variety forms as it comes to life in classrooms.

Teachers must bring to the classroom a broad repertoire of instructional and assessment strategies. In particular, teachers need to understand how to engage students in various types of learning activities that are appropriate for both the students and the subject matter. How do you set up a productive cooperative learning experience? How do you facilitate a discussion or debate? What are the components of an effective guided exploration? How can you tell when a student truly understands a concept and when a student is merely mimicking an observed pattern or procedure? When and how do you help students make a transition from their own language or representations to more standardized terms and notations?

Assessment and instruction are often interwoven when making classroom decisions. To plan effectively, teachers must know how to assess their students' prior knowledge, how to determine what their students should learn next, and how they can intellectually challenge their students. Well-designed lessons afford teachers opportunities to learn about their students' understandings and encourage students to refine their understandings to accommodate new ideas.

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To develop a motivational learning environment, teachers must understand what motivates their students. In particular, teachers must be familiar with their students' interests and abilities so they can help students see how mathematics emerges within those contexts. Similarly, when students are motivated by explorations using technology, manipulatives, or extended projects, teachers should find meaningful ways to engage students in those types of activities. Because students are willing to do what they believe they can do, teachers need to design learning experiences with various entry points so that students from diverse backgrounds can become engaged in the mathematics. Likewise, teachers must know how to select problems that are challenging enough to be interesting, yet not overwhelming, for their students.

Teachers should be knowledgeable of various verbal and nonverbal modes of communication. In particular, knowledge of technological applications can enhance a teacher's communication capabilities, providing a vehicle for dynamic graphical, numerical, and visual representations of mathematics concepts and skills. Likewise, hands-on and virtual manipulatives are powerful tools for representing mathematical concepts, thereby adding to a toolbox of representations from which students can communicate their mathematical understandings.

Standard 1: Knowledge of Mathematics and General Pedagogy

Vignette

In the first vignette, the teacher uses an exploratory, calculator-based task to spark students' mathematical thinking. While implementing the task, the teacher encourages her students to take intellectual risks by generating their own questions. As a result, the students generate ideas that relate to higher-level mathematics concepts. The teacher must draw on her own knowledge of those concepts in deciding how to respond to students' conjectures.

1.1—*Drawing on Mathematical Knowledge during Exploration Activities*

After recently completing a unit on multiplication and division, a fourth-grade class has just begun to learn about factors and multiples. Their teacher is having students use calculators as a tool for exploring that topic.

Using the automatic constant feature of their calculators (that is, that pressing 5 + = = = ... yields 5, 10, 15, 20, ... on the display), the fourth graders have generated lists of the multiples of different numbers. They have also used the calculator to explore the factors of different numbers. To encourage the students to deepen their understanding of numbers, the teacher has urged them to look for patterns and to make conjectures. She asked them, "Do you see any patterns in the lists you are making? Can you make any guesses about any of those patterns?"

Two students have raised a question that has attracted the interest of the whole class:

Are there more multiples of 3 or more multiples of 8?

The teacher encourages them to pursue the question, for she sees that it can engage them in the concept of multiples as well as provide a fruitful context for making mathematical arguments. She realizes that the question holds rich mathematical potential and even brings up questions about orders of infinity. "What do the rest of you think?" she asks. "How could you investigate this question? Go ahead and work on this a bit on your own or with a partner, and then let's discuss what you come up with."

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The children pursue the question excitedly. The calculators are useful once more as the students generate lists of the multiples of 3 and the multiples of 8. Groups are forming around particular arguments. One group of children argues that there are more multiples of 3 because in the interval between 0 and 20 there are more multiples of 3 than multiples of 8. Another group is convinced that the multiples of 3 are "just as many as the multiples of 8 because they go on forever." A few children think there should be more multiples of 8 because 8 is greater than 3.

Although the teacher intends to revisit the students' conjecture another time, she redirects the conversation. She asks students whether they think the size of the number relates to how many factors it has. Some students excitedly form a new conjecture about factors: The larger the number, the more factors it has.

The teacher is pleased with the ways in which opportunities for mathematical reasoning are growing out of the initial exploration. The question asked by the two students has promoted mathematical reasoning, eliciting at least three competing and, to fourth graders, compelling mathematical arguments. The teacher likes the way in which students are making connections between multiples and factors. She also notes that students already seem quite fluent using the terms *multiple* and *factor*.

Although the class period is nearing its end, the teacher invites one group to present to the rest of the class their conjecture that the larger the number, the more factors it has. She suggests that the students record the conjecture in their notebooks and discuss it in class tomorrow. Pausing for a moment before she sends them out to recess, she decides to provoke their thinking a bit more: "That's an interesting conjecture. Let's just think about it for a second. How many factors does, say, 3 have?"

"Two," call out several students.

"What are they?" she probes. "Yes, Deng?"

Deng quickly replies, "1 and 3."

"Let's try another one," continues the teacher. "What about 20?"

After a moment, several hands shoot up. She pauses to allow students to think, and asks, "Natasha?"

"Six: 1 and 20, 2 and 10, 4 and 5," answers Natasha with confidence.

The teacher suggests a couple more numbers, 9 and 15. She is conscious of trying to use only numbers that fit the conjecture. With satisfaction, she notes that most of the students are quickly able to produce all the factors for each of the numbers she gives them. Some used paper and pencil, some used calculators, and some used a combination of both. As she looks up at the clock, one child asks, "But what about 17? It doesn't seem to work."

"That's one of the things that you could examine for tomorrow. I want all of you to see if you can find out whether this conjecture always holds."

"I don't think it'll work for odd numbers," says one child.

"Check into it," smiles the teacher. "We'll discuss it tomorrow."

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Vignette

In the second vignette, the teacher uses his knowledge of assessment techniques to diagnose student errors. The teacher realizes that asking students to explain their work may give him better insight into the children's thinking. As a result of his investigation, the teacher is able to develop a strategy to help his students connect their understanding of subtraction concepts with their paper-and-pencil work.

1.2—*Using Student Interviews to Identify Misconceptions*

The second graders have just finished working on addition and subtraction with regrouping. Although the students have been using a variety of models and written algorithms for subtraction, their teacher, Mr. Lewis, notices that many of them seem to "forget" to regroup in subtraction when using the traditional algorithm. Instead, they sometimes do this:

$$\begin{array}{r} 50 \\ -\,38 \\ \hline 28 \end{array}$$

Mr. Lewis wonders whether the students are being careless or whether something else is going on. He decides to sit down with the children one by one for a few minutes and have them talk through a couple of the problems and how they solved them. He thinks he may be better able to follow their thinking if they explain their steps as they work through a problem.

The teacher decides to use manipulative materials to gather some additional information about the students' understanding. He chooses a couple of problems from the test and asks the children to justify their answers using bundles of craft sticks. He discovers that most of them are not connecting the manipulatives work they did in class with their work on the written problems. When they use the craft sticks, they find that their paper-pencil answers do not make sense, and they revise them to match what they did with the sticks.

Mr. Lewis had assumed that if the students "saw" the concepts by actually touching the objects, they would understand. He now thinks that maybe he did not do enough to help them build the links between the concrete model and the algorithm. He starts wondering what he could do to make that connection clearer. He hypothesizes that perhaps the students know how to regroup but may not understand why we sometimes regroup or when regrouping is necessary. As a result of talking individually with students, Mr. Lewis concludes that he should revise his instructional plan to help students discover *why*, *when*, and *how* they should perform subtraction with regrouping. He decides to make up a set of examples in which regrouping is necessary for some and not for others. He plans to have the children discuss whether they would regroup in each example and how they would decide.

Closing Thoughts
Standard 1: Knowledge of Mathematics and General Pedagogy

Teachers must have a deep knowledge of mathematics on which to base their instructional decisions. However, knowledge of mathematical content alone is not sufficient to prepare teachers for the many challenges they face in the classroom. Knowledge of students' development, modes of instruction, and effective motivational and communication techniques all serve as resources for effective teaching practice.

Standard 2: Knowledge of Student Mathematical Learning

Teachers of mathematics must know and recognize the importance of—

- what is known about the ways students learn mathematics;
- methods of supporting students as they struggle to make sense of mathematical concepts and procedures;
- ways to help students build on informal mathematical understandings;
- a variety of tools for use in mathematical investigation and the benefits and limitations of those tools; and
- ways to stimulate engagement and guide the exploration of the mathematical processes of problem solving, reasoning and proof, communication, connections, and representations.

Elaboration

Effective mathematics teachers create opportunities for students to develop their mathematical understandings, competence, and interests (Lampert 2001). They actively engage students in tasks that enable them to see mathematics as a coherent and connected endeavor rather than as a series of disconnected rules and procedures that they must memorize. To support that kind of teaching, teachers need mathematics-specific pedagogical knowledge: knowledge of the important ideas central to their grade level, about representations that can most effectively be used in teaching those ideas, about the common misconceptions in learning those ideas, and about designing lessons that actively involve students in building their mathematical understanding (Shulman 1987).

Teachers rely on their pedagogical knowledge to make decisions about what to teach, when to teach it, and how to teach it. Specifically, teachers know how to assess their students' existing knowledge by carefully listening to students' explanations, by observing their manipulations of learning tools, and by analyzing their written work. One of the most valuable ways to identify and extend students' existing knowledge is to make use of problems that are set in real-world contexts with which students are familiar. In such settings, teachers can assess students' existing understandings as well as help students see the value of the mathematics they are learning. Teachers then draw on their knowledge and experience to connect with students intellectually and to help students revisit and refine their understandings. Mathematics learning is not a one-size-fits-all enterprise. Expert teachers know how to tailor experiences to fit the needs of individual students.

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Teachers must know how to construct and sequence questions that engage students in the activities and help them focus on relevant aspects of the mathematics.

Mathematical ideas cannot be learned in a vacuum. Teaching mathematical topics as disconnected entities, or as a sequence of "tricks of the day," may lead to high quiz scores at the end of the week but rarely will lead to long-term understanding (Steen 1999). Rather, in-depth explorations of the relationships among representations and ideas help develop a more reliable and sustainable capacity to use, transfer, and understand mathematical ideas and procedures. Guiding such explorations requires extensive preparation and knowledge. Teachers must know how to construct and sequence questions that engage students in the activities and help them focus on relevant aspects of the mathematics. Teachers must realize when students have misinterpreted a situation and be prepared with questions that will help students discover their errors and rethink their strategies.

As students explore, investigate, and consider mathematical ideas, knowing when to interrupt and when to let students wrestle with ideas are important considerations for teachers. Although well-formulated solutions and clear explanations may be good indicators of understanding, merely listening to someone else's well-articulated ideas does not necessarily assure that the listener will understand or be able to offer a clear explanation when asked. As a result, teachers must be able to recognize appropriate moments to intervene as well as times when the students will benefit more by resolving challenges on their own (Hiebert and Wearne 2003).

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Lessons that enable students to make connections among concrete, graphic, symbolic, and verbal representations often incorporate manipulatives, technology, and open-ended activities. Such lessons require students to communicate using the language of mathematics as they reason through and solve problems. Because the lessons provide the stimulus for students to think about particular concepts and procedures, their connections with other mathematical ideas, and their applications to real-world situations, such lessons can help students to develop skills in contexts in which they will be useful. Teachers must have at their disposal, and be well versed in using, a variety of tools for generating, comparing, and connecting multiple representations. Teachers must select appropriate tools for the specific content at hand, must know how to exploit the conceptual advantages inherent in each tool, and must know how to address or avoid a tool's potential weaknesses—all this within the context of that teacher's classroom of students.

Although students should engage in a variety of mathematical processes throughout school, those processes may look very different in early elementary grades than they do in high school. As a result, teachers must know how to establish standards of reasoning that are appropriate for their own students and know how those standards may differ from what counts as a complete justification at other grade levels. Likewise, to effectively orchestrate a class discussion during which students share a variety of solution strategies, teachers must be aware of the most common strategies used to solve problems as well as the distinctive elements and connections among them.

Standard 2: Knowledge of Student Mathematical Learning

Vignette

In the first vignette, the teacher uses her knowledge of how children learn mathematics to develop a lesson that emphasizes conceptual understanding of division by a fraction. The teacher draws from existing curricular resources and modifies the tasks to better suit her pedagogical purposes.

2.1—*Modifying Resources to Meet Students' Needs*

Ms. Pierce is a third-year teacher in a large middle school. She uses a mathematics textbook, published about ten years ago, that her department strongly suggests she closely follow. In the midst of a unit on fractions with her seventh graders, Ms. Pierce is examining her textbook's treatment of division with fractions. Her previous two years of teaching experience and her teacher preparation background have caused her to be reflective in analyzing the goals and intent of textbook lessons. She is trying to determine this lesson's strengths and weaknesses and whether and how she should use the textbook as written to help her students understand division with fractions.

She notices that the textbook's emphasis is on the mechanics of carrying out the procedure ("dividing by a number is the same as multiplying by its reciprocal"). The text mentions that students "can use reciprocals to help" them divide by fractions and shows a few examples of the procedure. However, Ms. Pierce wants her students to understand what it means to divide by a fraction, not *simply* learn the mechanics of the procedure.

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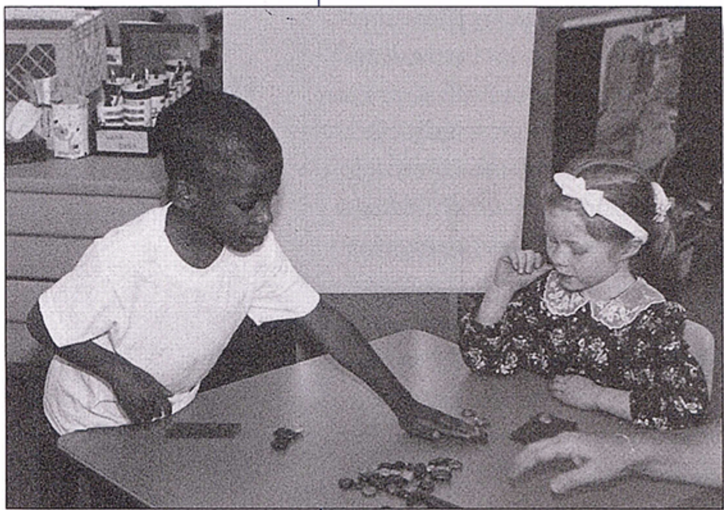
The picture at the top of one of the textbook pages shows some beads of a necklace lined up next to a ruler. This graphic is an attempt to represent that twenty-four 3/4-inch beads and forty-eight 3/8-inch beads are in an eighteen-inch necklace. Ms. Pierce notes that the graphic does represent what is meant by dividing by 3/4 or by 3/8, specifically related to the question "How many three-fourths or three-eighths are there in eighteen?" Still, when she considers what would help her students understand the problem, she does not think that this particular representation is adequate. She also suspects that students may not take the representation seriously, for they too often tend to perceive mathematics as memorizing rules rather than a means for understanding why the rules work.

Ms. Pierce senses that the idea of "using the reciprocal" is introduced almost as a trick, lacking any rationale or connection with the pictures of necklaces. Furthermore, division with fractions seems to be presented as a new topic, not connected with anything that the students might already know, such as division of whole numbers. Ms. Pierce is concerned that the textbook pages are likely to reinforce that impression. She does not see anything in the task that would emphasize the value of understanding why the procedure works, nor that would promote mathematical discourse.

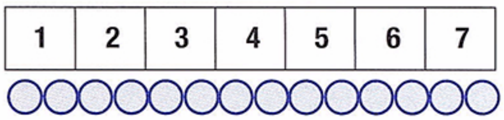


Photograph by Susan Callum; all rights reserved

Thinking about her students, Ms. Pierce judges that the two pages under discussion require computational skills that most of her students already possess (i.e., being able to produce the reciprocal of a number and being able to multiply fractions) but that the exercises on the pages would not be interesting to them. Little on the two text pages would engage their thinking.

Looking at the pictures of the necklaces gives Ms. Pierce an idea. She decides she can use this idea, so she copies only the drawing. The necklace model is linear rather than circular, as are diagrams of a pie or pizza that are most often used to represent fractions. Ms. Pierce uses the linear model to help students develop varied representations. She also realizes that different representations make sense to different students.

To help students see connections between whole-number division and rational number division, Ms Pierce uses division models that can be used with whole numbers, rational numbers, or a combination of the two. She plans to include at least one picture that shows beads of whole-number length, for example, 2-inch beads (not shown here), before she shows pictures with beads of rational-number length, such as the ½-inch beads shown in the diagram. She will ask students to examine the pictures and try to write some kind of number sentence that represents what they see. For example, this 7-inch bracelet has fourteen 1/2-inch beads:



This situation could be represented as either $7 \div 1/2$ or 7×2. She will try to help students think about the reciprocal relationship between multiplication and division and the meaning of dividing something by a fraction or by a whole number. Then she thinks she could use some of the computational exercises on the second page of the textbook, but instead of just having the students compute the answers, she will ask them, in pairs, to write stories for each of about five exercises. Ms. Pierce knows that writing stories to go with the division sentences may help students focus on the meaning of the procedure.

She decides she will also provide a couple of other examples that involve whole-number divisors: $28 \div 8$ and $80 \div 16$, for example.

Ms. Pierce feels encouraged from her experience with planning this lesson and thinks that revising other textbook lessons will be feasible. Despite the fact that she is supposed to be following the text closely, Ms. Pierce now knows that she will be able to do so but still be able to adapt the text in ways that will significantly improve what she can do with her students this year.

Vignette

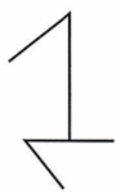
In the following vignette, a teacher assesses students' existing knowledge as the basis for a lesson in which students explore mathematical definitions and relationships among types of figures. Teacher-generated counterexamples serve as a powerful tool for helping students refine their language as they develop the components of a mathematical definition. When the students are ready, the teacher facilitates a discussion of how to further discriminate among figures that fit the definition.

2.2—*Helping Students Build on Informal Understandings*

Mrs. Logan is beginning an exploration of the properties of quadrilaterals for the purpose of creating a definition. She opens class by eliciting students' ideas, asking simply, "What do you know about quadrilaterals? How would you describe a quadrilateral?"

Several students respond in unison, "four sides," "four-sided figure."

Mrs. Logan reacts to exactly what the students say by making sketches that match students' descriptions. For example, Mrs. Logan draws



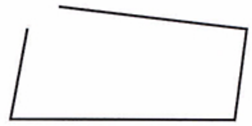
and asks, "Is this a quadrilateral?"

Students: No, it has to connect.

Mrs. Logan: Is this one?

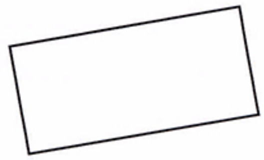
Several students: No, it can't intersect like that.

Mrs. Logan continues drawing and asks, "So is this one?"



Student: It has to close.

Mrs. Logan: Okay, then is this one?



Students: Yes!

The teacher next asks a question designed to prompt students to identify specific characteristics of quadrilaterals on the basis of their discussion. Mrs. Logan pauses and then looks directly at the students: "I drew four examples. You said three of those didn't work. Can you explain what makes the difference?"

Several students volunteer elements that may become part of a definition of *quadrilateral*. Mrs. Logan lists their ideas—in their terms—on the board:

Quadrilaterals

4 points

4 segments

No more points intersect

Closed curve

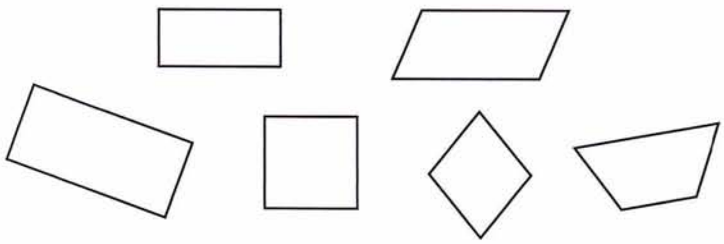
Summarizing, Mrs. Logan tells them, "You really seem to have all the pieces. The definition of quadrilateral in our text is 'the union of four segments joined at their endpoints such that the segments intersect only at the endpoints.' Now, which of our figures fit with this definition?"

After a brief discussion during which they analyze their figures, Mrs. Logan continues, "And what are some special kinds of four-sided figures?"

Students call out a variety of names: *square, rectangle, kite, rhombus, parallelogram.*

Mrs. Logan: Another one you should have heard of before is *trapezoid.*

She draws several figures on the board, and asks, "Can you classify these and talk about them?"



Students begin murmuring, in pairs and groups of three.

Mrs. Logan: Are there other special quadrilaterals that are not shown here?

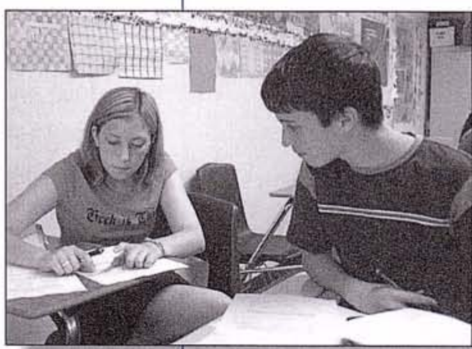
One usually reticent student asks, "But isn't a square also a rectangle? I don't quite see how to classify these."

Instead of responding directly to his question, Mrs. Logan decides it is worth everyone's consideration and redirects the question to the group.

Mrs. Logan replies, "Why don't you put that to the rest of the class? See what they think."

The student repeats his question to classmates.

Mrs. Logan adds, "See if you can find a way to classify these shapes. Which shapes have which labels?"



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The students resume their work. Some students begin making diagrams to represent the interrelationships among the types of quadrilaterals while others begin making tables. Mrs. Logan walks around, questioning students to get them to clarify what they are thinking.

Mrs. Logan asks one boy to explain his rather complicated chart. "And why is the rhombus there, with the parallelogram?"

After class, Mrs. Logan contemplates the lesson. In general, she thinks it was a good start. Perhaps tomorrow—to help the students begin to construct some of the categories—she will engage them in a problem or activity in which they will have to sort quadrilaterals. She muses a bit about how to frame it in a way that will promote discourse and student understanding.

Closing Thoughts
Standard 2: Knowledge of Student Mathematical Learning

Beyond mathematics and general pedagogy, teachers need to draw on mathematics-specific pedagogy to best help their students develop both conceptual understanding and

The cycle of planning, implementing, and reflecting contributes both to students' learning and to teachers' continual improvement.

procedural proficiency. In particular, mathematics teachers must know how to actively engage their students in the mathematical processes of problem solving, reasoning, representing, communicating, and making connections. The cycle of planning, implementing, and reflecting contributes both to students' learning and to teachers' continual improvement.

Summary of the Knowledge Standards

The Teaching and Learning Principles of *Principles and Standards for School Mathematics* set goals for a new vision of the mathematics classroom. Departing from traditional practices, teachers use their knowledge of mathematics, pedagogy, and their students to develop lessons that foster students' learning and engagement. They assume that all students can learn and use significant mathematics, and they craft lessons to involve and challenge all students. How teachers implement their knowledge in the classroom is examined in the next section.

Creating an environment that supports and encourages mathematical reasoning and fosters all students' competence with, and inclination toward, mathematics should be one of a teacher's central concerns.

Implementation

Teachers must implement their knowledge about the teaching and learning of mathematics by choosing challenging tasks and facilitating meaningful mathematical discourse within a healthy and supportive learning environment. Creating an environment that supports and encourages mathematical reasoning and fosters all students' competence with, and inclination toward, mathematics should be one of a teacher's central concerns. The nature of such a learning environment is shaped by the kinds of mathematical tasks and discourse in which students engage. The tasks must encourage students to reason about mathematical ideas, to make connections, and to formulate, grapple with, and solve problems. In practice, students' actual opportunities for learning depend, to a considerable degree, on the kind of discourse that the teacher orchestrates. The nature of the activity and talk in the classroom shapes each student's opportunities to learn about particular topics as well as to develop her or his abilities to reason and communicate about those topics (Henningsen and Stein 1997).

Standard 3: Worthwhile Mathematical Tasks

The teacher of mathematics should design learning experiences and pose tasks that are based on sound and significant mathematics and that—

- engage students' intellect;
- develop mathematical understandings and skills;
- stimulate students to make connections and develop a coherent framework for mathematical ideas;