$$2n + 2(n + 3)$$
$$(n + 2) + (n + 2) + (n + 1) + (n + 1)$$
$$4n + 2 + 4$$
$$2(2n + 1) + 4$$
$$(n + 2)(n + 3) - (n)(n + 1)$$

Mr. Rodriquez is pleased with the variety of expressions that are generated. He asks the class, "These all seem different. How can they all be correct? Maybe we should start by having you explain where these came from so we can see if they make sense. I'm especially intrigued by the last one. Bonnie, can you tell us how your group came up with that?"

Bonnie: We thought about the frame as a rectangle taken away from the middle of a bigger rectangle. So the outer rectangle has dimensions $(n + 3)$ by $(n + 2)$, and the inner rectangle has dimensions (n) by $(n + 1)$. Then we just subtracted the inner area from the outer area.

After discussing the rest of the student-generated expressions, Mr. Rodriguez plans to give the class another pattern of blocks and ask them a similar series of questions. He hopes that they will move to a generalization more quickly so that he can begin to ask questions that will help students focus on the concept of equivalence of algebraic expressions.

When reflecting on this class session, the teacher concludes that the exploration was worthwhile because all the students were quickly engaged in the task, it required that students give meaning to variables and expressions, and a valuable class discussion ensued.

Closing Thoughts
Standard 3: Worthwhile Mathematical Tasks

To engage students in challenging mathematics, teachers need to use worthwhile mathematical tasks. Those tasks must be rich in terms of content and processes. Tasks that promote communication and connections can help students see and articulate the value and beauty of mathematics. Teachers can enhance the value of existing materials by tailoring them to needs and interests of their students. By doing so, teachers can promote both motivation and equity in their classrooms.

Tasks that promote communication and connections can help students see and articulate the value and beauty of mathematics.

Standard 4: Learning Environment

The teacher of mathematics should create a learning environment that provides—

- the time necessary to explore sound mathematics and deal with significant ideas and problems;

- a physical space and appropriate materials that facilitate students' learning of mathematics;
- access and encouragement to use appropriate technology;
- a context that encourages the development of mathematical skill and proficiency;
- an atmosphere of respect and value for students' ideas and ways of thinking,
- an opportunity to work independently or collaboratively to make sense of mathematics;
- a climate for students to take intellectual risks in raising questions and formulating conjectures; and
- encouragement for the student to display a sense of mathematical competence by validating and supporting ideas with a mathematical argument.

Elaboration

The mathematics teacher is responsible for creating an intellectual environment in which serious engagement in mathematical thinking is the norm. More than just a physical setting with desks, bulletin boards, and posters, the classroom environment is suggestive of a hidden curriculum with messages about what counts in learning and doing mathematics. If teachers want students to learn to make conjectures, experiment with alternative approaches to solving problems, and construct and respond to others' mathematical arguments, then creating an environment that fosters those kinds of activities is essential (NCTM 2000).

A central focus of the classroom environment must be sense making.

A central focus of the classroom environment must be sense making (Hiebert et al. 1997). Mathematical concepts and procedures—indeed, mathematical skills—are central to making sense of mathematics and to reasoning mathematically. Teachers should consistently expect students to explain their ideas, to justify their solutions, and to persevere when they encounter difficulties. Teachers must also help students learn to expect and ask for justifications and explanations from one another. A teacher's own explanations must similarly focus on underlying meanings; something a teacher says is not necessarily true simply because he or she "said so."

Emphasizing reasoning and justification implies that students should be encouraged and expected to question one another's ideas and to explain and support their own ideas in the face of challenges from others (Lampert 2001). Teachers must assist students in learning how to do so. Teachers must create a classroom environment in which students' ideas, whether conventional or nonstandard, whether valid or invalid, are valued. Students need to learn how to justify their claims without becoming hostile or defensive. As teachers help students respect and show interest in the ideas of their classmates, students will be more likely to take risks in proposing their conjectures, strategies, and solutions.

Serious mathematical thinking takes time as well as intellectual courage and skills. A learning environment that supports problem solving must provide time for students to puzzle, to be stuck, to try alternative approaches, and to confer with one another and with the teacher. Furthermore, for many worthwhile mathematical tasks, especially those that require reasoning and problem solving, the speed, pace, and quantity of students' work are inappropriate criteria for "doing well." Too often, students have developed the belief that, if they cannot answer a mathematical question almost immediately, then they might as well give up. Teachers must encourage and expect students to persevere when they encounter mathematical challenges and invest the time required to figure things out. In discussions, the teacher must allow time for students to respond to questions and must also expect students to give one another time to think without interrupting or showing impatience (NCTM 2000).

Teachers must encourage and expect students to persevere when they encounter mathematical challenges and invest the time required to figure things out.

Students' learning of mathematics is enhanced in a learning environment that is a community of people collaborating to make sense of mathematical ideas (Hiebert et al. 1997). A significant role of the teacher is to develop and nurture students' abilities to learn with and from others—to clarify individual interpretations of definitions and terms with one another, to consider one another's ideas and solutions, and to argue together about the validity of alternative approaches and solutions. Various classroom structures can encourage and support such collaboration. Students may at times work independently, conferring with others as necessary; at other times students may work in pairs or in small groups. Students may use the Internet to research and collect data, use interactive geometry software to conduct investigations, or use graphing calculators to translate among different mathematical representations. Whole-class discussions are yet another profitable format. No single arrangement will work at all times; teachers should use various arrangements and tools flexibly to pursue their goals.

Students' learning of mathematics is enhanced in a learning environment that is a community of people collaborating to make sense of mathematical ideas.

Standard 4: Learning Environment

Vignette

The first vignette illustrates a learning environment in which students work independently and collaboratively, take responsibility for their own and one another's learning, and feel comfortable sharing ideas publicly. This environment fosters the development of student understanding about concepts and the connections among them.

4.1—*The Role of the Environment in Supporting Student Development*

Ms. Chavez is using a wireless learning system along with graphing calculators in her class and has her calculator connected to her LCD viewer. Her twenty-eight first-year algebra students, seated at round tables in groups of threes and fours, are working on a warm-up problem. The day before, they had had a test on functions. For the warm-up to today's class, Ms. Chavez has asked students to set up a table of values and use the table to create a graph of the function $y = |x|$.

She has chosen this problem as a way to introduce some ideas for a new unit on linear, absolute value, and quadratic functions. During the warm-up, students can be heard talking quietly to one another about the problem: "Does your graph look like a V-shape?" "Did you get two intersecting lines?" Walking around the room, Ms. Chavez listens to the students' conversations and uses her wireless system to capture the calculator screens and the work of students so she can get a feel for their attempts. After about five minutes, she signals that the time has come to begin the whole-group discussion.

A girl volunteers and carefully sketches her graph on a large dry-erase grid at the front of the room while the teacher displays the student's table of values using the LCD projector. As she does so, most students are watching closely, glancing down at their own graphs and tables, checking for correspondence. In this class, students are expected to communicate about mathematics. They also accept responsibility for helping others.

Another student suggests that they project Elena's graph, and the class watches as the graph appears on the projector screen. It matches the graph sketched by the first student, and the class cheers, "Way to go, Elena!"

Ms. Chavez then asks the class to sketch the graphs of $y = |x| + 1$, $y = |x| + 2$, and $y = |x| - 3$ on the same set of axes and write a paragraph that compares and contrasts the results with the graph of $y = |x|$. "Feel free to work alone or with the others in your group," she tells them.

After a few minutes, two students exclaim, "All the graphs have the same shape!"

A few other students look up. Another student observes, "They're like angles with different vertex points." "Then they're really congruent angles," adds his partner.

Ms. Chavez circulates through the classroom, listening to the discussions, asking questions, and offering suggestions. One group asks, "What would happen if we tried $|x - 3|$?" "Try it!" urges Ms. Chavez.

The students continue working, and the conversation is lowered to murmurs once again. Then the members of one group call out, "Hey, we've got something! All these graphs are just translations of $y = |x|$, just like we learned in the unit on geometry."

Ms. Chavez is pleased that her students seem to be making connections between this graphing activity and transformational geometry. "That's an interesting conjecture you have," remarks Ms. Chavez. She looks expectantly at the other students. "Do the rest of you agree?" They are quiet, many looking hard at their graphs. One student says, slowly, "I'm not sure I get it." Ms. Chavez has the students send their graphs, and all graphs are simultaneously displayed via the LCD projector. "Now, what do you think?" she asks.

A boy in the group that made the conjecture about translations explains, "Like, $y = |x| + 2$ is like $y = |x|$ moved up two spaces, and $y = |x| - 3$ is moved down three spaces. It's like what Louella said about them being like angles with different vertex points."

Ms. Chavez decides to prompt the class to pursue his idea. She asks whether anyone can graph $y = |x| + 4$ without first setting up a table of values and without using calculators. Hands shoot up. "Ooooh!" Scanning the class, Ms. Chavez notices that Lionel, who does not volunteer often, has his hand up. He looks pleased when she invites him to give it a try.

Lionel sketches his graph on the dry-erase board. Elena again enters the equation of the graph into the calculator, and the class watches as the graph is produced. The calculator-generated graph verifies Lionel's attempt. Again cheers erupt. Lionel gives a sweeping bow and sits down.

Ms. Chavez asks the students to write in their journals, focusing on what they think they understand and what they feel unsure about from today's lesson. Their journals give the teacher insights into students' thinking. They also offer students the opportunity to reflect on their understandings and feelings.

At the end of the period, Ms. Chavez distributes the homework that she has prepared. The worksheet includes additional practice on the concept of $y = |x| \pm c$ as well as something new, to provoke the next day's discussion: $y = |x \pm c|$.

Vignette

In the second vignette, students work together to solve problems. Sometimes they build on the solutions offered by classmates. The teacher has created an environment in which students expect to have to justify their solutions, not just give answers.

The teacher has created an environment in which students expect to have to justify their solutions, not just give answers.

4.2—Encouraging Sense Making by Expecting Students to Reason

A class of primary students has been working on problems that involve separating or dividing. The teacher, Laurie Morgan, is trying to give them some early experience with multiplicative situations at the same time that she provides them with contexts for deepening their knowledge of, and skill with, addition and subtraction. The students can add and subtract, but their understanding of multiplication and division is still quite informal. They have begun to develop some understanding of fractions, connected with their ideas about division. They have not yet learned any conventional procedures for dividing.

Today Ms. Morgan has given them the following problem:

If we make 49 sandwiches for our picnic, how many can each child have?

The teacher has selected this problem because it is likely to elicit alternative representations and solution strategies as well as different answers. It will also help the students develop their ideas about division, fractions, and the connections between them.

After they have worked for about ten minutes, first alone and then in small groups, Ms. Morgan asks whether the children are ready to discuss the problem in the whole group. Most, looking up when she asks, nod. She asks who would like to begin.

$$\begin{array}{r} 49 \\ -\ 28 \\ \hline 22 \end{array}$$

Two girls go to the white board. They write the following problem:

One explains, "There are twenty-eight kids in our class, and so if we pass out one sandwich to each child, we will have twenty-two sandwiches left, and that's not enough for each of us, so there'll be leftovers."

The teacher and students are quiet for a moment, thinking about the proposed solution. Ms. Morgan looks over the group and asks whether anyone has a comment or a question about the solution. She expects the students, as members of a learning community, to decide whether an idea makes sense mathematically.

One boy says that he thinks their solution makes sense but that "9 minus 8 is 1, not 2, so it should be 21, not 22." He demonstrates by pointing at the number line above the white board. Starting at 9, he counts back eight using a pointer. The two girls ponder this demonstration for a moment. The class is quiet. Then one says, "We revise that. Nine minus 8 is 1." Ms. Morgan is listening closely but does not jump into the interchange. She wants students to respectfully question one another's ideas.

Another child remarks that he had the same solution as they did—one sandwich.

"Frankie?" asks Ms. Morgan, after pausing for a moment to look over the students. She remembers noticing his approach during the small-group time. Frankie announces, "I think we can give each child more than one sandwich. Look!" He proceeds to draw twenty-one rectangles on the white board. "These are the leftover sandwiches," he explains. "I can cut fourteen of them in half, and that will give us twenty-eight half-sandwiches, so everyone can get another half."

"I agree with Frankie," says another child. "Each child can have one and a half sandwiches."

"Do you have any leftovers?" asks the teacher.

"There are still seven sandwiches left over," says Frankie.

"What do the rest of you think about that?" inquires the teacher.

Several children give explanations in support of Frankie's solution. "I think that does make sense," says one girl, "but I had another solution. I think the answer is 1 plus 1/2 plus 1/4."

"I don't understand," Ms. Morgan says. "Could you show what you mean?"

Ms. Morgan encourages her students to take risks by bringing up different ideas. But raising questions is not enough. She also reminds students that they are expected to clarify and justify their ideas.

Closing Thoughts
Standard 4: Learning Environment

To capitalize on the value of worthwhile mathematical tasks, teachers and students must explore those tasks within a classroom environment that fosters intellectual growth and development. In such an environment, an objective of both teachers and students is to use mathematical tools and skills to make sense of problems, patterns, and contradictions. Participants use appropriate technology in such an environment to foster insight into mathematical situations. Students work independently or collaboratively to develop skills, make conjectures, and develop arguments within a mathematical community that values the contributions of all participants and defers to the authority of sound reasoning in the search for mathematical truth.

Standard 5: Discourse

The teacher of mathematics should orchestrate discourse by—

- posing questions and tasks that elicit, engage, and challenge each student's thinking;
- listening carefully to students' ideas and deciding what to pursue in depth from among the ideas that students generate during a discussion;
- asking students to clarify and justify their ideas orally and in writing and by accepting a variety of presentation modes;
- deciding when and how to attach mathematical notation and language to students' ideas;
- encouraging and accepting the use of multiple representations;
- making available tools for exploration and analysis;
- deciding when to provide information, when to clarify an issue, when to model, when to lead, and when to let students wrestle with a difficulty; and
- monitoring students' participation in discussions and deciding when and how to encourage each student to participate.